Bootstrapping Bipower Variation

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Abstract

In this paper we extend two bootstrap methods, primarily introduced by Gonçalves and Meddahi (2004) in context of realized volatility. We consider the i.i.d. bootstrap and the wild bootstrap and prove their first-order asymptotic validity for more general statistics than the realized volatility, namely for bipower variation. In addition to that we use Edgeworth expansions and Monte Carlo simulations to compare the accuracy of the bootstrap with other existing approaches. It is shown that the wild bootstrap provides a second-order asymptotic refinement, if the external random variable is chosen appropriately and the volatility is constant. The i.i.d bootstrap has the same rate of convergence as the error implied by the standard normal distribution for constant volatility. Nevertheless, Monte Carlo simulations suggest that both methods improve upon the first-order asymptotic theory in finite samples.

1 Introduction

In mathematical and financial literature there is an increasing number of articles concerning the estimation and application of the integrated volatility and related quantities. This is based on the multitude and the importance of these applications (see, e.g., Anderson et al. (2006)). Moreover, the increasing availability of high frequency financial data leads to a rising popularity of realized volatility as a measure of volatility in empirical finance. The realized volatility is, under weak conditions, a consistent estimator of the integrated volatility (see, e.g., Andersen et al. (2002)). Jacod (1994), Jacod and Protter (1998) and Barndorff-Nielsen and Shephard (2002)[1] developed a theory for the asymptotic behaviour of measures similar to the realized volatility.

Furthermore, they derived central limit theorems for the realized volatility and related estimators like realized bipower or multipower variation (see, e.g., BN-S (2003, 2004a)). This is used to test for the presence of jumps in asset prices (BN-S (2006) or Christensen and Podolskij (2006a)) or to calculate confidence sets for the estimators.

Recently Gonçalves and Meddahi (2004) presented bootstrap methods for realized volatility. They proved their first-order asymptotic validity and used Edgeworth expansions and Monte Carlo simulations to compare the accuracy of the bootstrap with the existing first-order feasible theory. Monte Carlo simulations suggest that the bootstrap methods improve upon the first-order asymptotic theory in finite samples.

[1] Henceforth BN-S.

In this paper we analyze these two bootstrap methods for bipower variation. The i.i.d. bootstrap (Efron (1979)) generates bootstrap pseudo intraday returns by resampling with replacement from the original set of intraday returns. With the wild bootstrap (Wu (1986)) intraday returns are generated by multiplying each original intraday return by an i.i.d. external random variable.
The remainder of this paper is organized as follows. In Section 2, we will introduce the basic model and the main assumptions. Furthermore, we review the existing first-order asymptotic theory. In Section 3, we introduce the i.i.d. bootstrap and prove the first-order asymptotic validity. In Section 4 we do the same for the wild bootstrap, while in Section 5 we discuss the second-order accuracy of both bootstrap methods. Finally, in Section 6, we construct confidence sets and discuss the finite sample properties. In the Appendix we give some technical results and present some simulation results in order to illustrate the finite sample properties of the proposed procedures.

2 Review

In finance models of the type

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s,\ t \in [0,1], \tag{1}$$

defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, are commonly used to describe the dynamics of assets. Here $W = (W_t)_{t\geq 0}$ denotes a one-dimensional Brownian motion, $b = (b_t)_{t\geq 0}$ is a locally bounded and predictable drift term and $\sigma = (\sigma_t)_{t\geq 0}$ is the volatility process, which is cádlág and nonnegative. Processes given by (1) are usually called Brownian semimartingale (BSM).
We assume that equidistant high frequency data on $[0,t]$ are observable. So we are in a position to define intraday returns for X_t on the time span $[\frac{i-1}{n}, \frac{i}{n}]$, i.e.

$$\Delta_i^n X = X_{\frac{i}{n}} - X_{\frac{i-1}{n}}. \tag{2}$$

2.1 Bipower variation

With the above notations we consider the realized bipower variation estimator (BN-S (2004a))

$$V(X,r,s)_t^n = n^{\frac{r+s}{2}-1} \sum_{i=1}^{\lfloor nt \rfloor -1} \mid \Delta_i^n X \mid^r \mid \Delta_{i+1}^n X \mid^s \qquad r,s \geq 0. \tag{3}$$

Furthermore, we define $\mu_r = E[\mid u \mid^r]$ and $u \sim N(0,1)$. A formula for μ_r is given by

$$\mu_r = \frac{2^{\frac{r}{2}} \Gamma(\frac{r+1}{2})}{\sqrt{\pi}}. \tag{4}$$

Bipower variation plays a central role in many applications in practice. One reason is its robustness to finite activity jumps if $r,s < 2$. Furthermore, it can be used to construct tests for the presence of jumps in asset prices (BN-S (2006) or Christensen and Podolskij (2006a)).
4,0-bipower, for example, is used to obtain feasible central limit theorems for estimates of integrated volatility (BN-S (2004b)).
Barndorff-Nielson, Graversen, Jacod, Podolskij and Shephard (2006) have shown the convergence in probability for the quantity $V(X,r,s)_t^n$, namely

$$V(X,r,s)_t^n \xrightarrow{\mathcal{P}} V(X,r,s)_t, \tag{5}$$

where $V(X,r,s)_t$ is given by

$$V(X,r,s)_t = \mu_r \mu_s \int_0^t \mid \sigma_u \mid^{r+s} du. \tag{6}$$

Remark 1
Two widely used estimators for the integrated variance $IV = V(X,2,0)_t = \int_0^t \sigma_u^2 du$ are 2,0-bipower and standardised 1,1-bipower, defined as

$$V(X,2,0)_t^n = \sum_{i=1}^{\lfloor nt \rfloor} \mid \Delta_i^n X \mid^2 \tag{7}$$

and

$$\mu_1^{-2} V(X,1,1)_t^n = \mu_1^{-2} \sum_{i=1}^{\lfloor nt \rfloor -1} \mid \Delta_i^n X \mid\mid \Delta_{i+1}^n X \mid, \tag{8}$$

respectively. In the presence of finite activity jumps 2,0-bipower becomes an inconsistent estimator for the integrated variance while standardised 1,1-bipower remains still consistent (BN-S (2006)).

With some additional assumptions it is possible to derive a central limit theorem for the above quantities. We still assume that $X_t \in BSM$. Furthermore, we assume that

(**V**) σ is everywhere invertible (V_1) and fulfills (V_2):

$$\sigma_t = \sigma_0 + \int_0^t \acute{b}_u \mathrm{du} + \int_0^t \acute{\sigma}_u \mathrm{dW}_u + \int_0^t \acute{v}_u \mathrm{dB}_u. \tag{9}$$

Here $\acute{b} = (\acute{b}_t)_{t\geq 0}$, $\acute{\sigma} = (\acute{\sigma}_t)_{t\geq 0}$ and $\acute{v} = (\acute{v}_t)_{t\geq 0}$ are adapted cádlág Processes. Furthermore, $\acute{b}$ is locally bounded and predictable and $B = (B_t)_{t\geq 0}$ is a Brownian motion independent of W. Besides these assumptions we need a special mode of convergence, namely stable convergence in law. To avoid confusions about terminology, we present the definition here, as it is probably not widely familiar to people in econometrics and finance.

Definition 1
A sequence of random variables, $\{X_n\}_{n\in\mathbb{N}}$, converges stably in law with limit X, defined on an appropriate extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, if and only if for every $\mathcal{F}$-measurable, bounded random variable Y and any bounded, continuous function g, the convergence $\lim_{n\to\infty} E[Yg(X_n)] = E[Yg(X)]$ holds.

We use the symbol $X_n \stackrel{\mathcal{D}_s}{\to} X$ to denote stable convergence in law (see, e.g., Renyi (1963) or Aldous and Eagleson (1978) for more details). Note that stable convergence implies weak convergence, which may equivalently be defined by taking Y=1. This stronger version is required to prove a standard CLT.

CLT for Bipower Variation

Let $r,s \geq 0$. Assume that (**V**) holds. Then we have for any $t > 0$

$$\sqrt{n}(V(X,r,s)_t^n - V(X,r,s)_t) \stackrel{\mathcal{D}_s}{\to} U(r,s)_t, \tag{10}$$

where $U(r,s)_t$ is defined by

$$U(r,s)_t = \sqrt{\mu_{2r}\mu_{2s} + 2\mu_r\mu_s\mu_{r+s} - 3\mu_r^2\mu_s^2} \int_0^t |\ \sigma_u\ |^{r+s}\, d\hat{W}_u, \tag{11}$$

and $\hat{W}$ is a 1-dimensional Brownian motion defined on an extension of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and independent of the σ-field $\mathcal{F}$.
As a consequence it is possible to state the following central limit theorems, which find a multitude of practical applications (see Barndorff-Nielsen, Graversen, Jacod, Shephard (2006))

$$S_n = \frac{\sqrt{n}(V(X,r,s)_t^n - V(X,r,s)_t)}{\rho(r,s)_t} \xrightarrow{\mathcal{D}} N(0,1) \tag{12}$$

and

$$T_n = \frac{\sqrt{n}(V(X,r,s)_t^n - V(X,r,s)_t)}{\rho(r,s)_t^n} \xrightarrow{\mathcal{D}} N(0,1). \tag{13}$$

The quantities $\rho^2(r,s)_t$ and $\rho^2(r,s)_t^n$ are given by

$$\rho^2(r,s)_t = (\mu_{2r}\mu_{2s} + 2\mu_r\mu_s\mu_{r+s} - 3\mu_r^2\mu_s^2) \int_0^t |\ \sigma_u\ |^{2(r+s)}\, du \tag{14}$$

and

$$\rho^2(r,s)_t^n = \frac{\mu_{2r}\mu_{2s} + 2\mu_r\mu_s\mu_{r+s} - 3\mu_r^2\mu_s^2}{\mu_{2r}\mu_{2s}} V(X,2r,2s)_t^n. \tag{15}$$

It is worthwhile to mention that $\rho^2(r,s)_t$ is just the conditional variance of the limiting process $U(r,s)_t$. Moreover, the convergence in (5) implies

$$\rho^2(r,s)_t^n \xrightarrow{\mathcal{P}} \rho^2(r,s)_t. \tag{16}$$

3 i.i.d. Bootstrap

The bootstrap (Efron (1979)) is a resampling method, which can be used for a multitude of problems in different ways. In the following we will introduce two bootstrap methods, which enable us to calculate confidence sets for bipower variation estimators. This is a generalization of the procedures introduced recently by Gonçalves and Meddahi (2004).
For the rest of the article we assume without loss of generalization that data on $[0,1]$ are observable. As $t = 1$ we omit the time index t in the following.
We define $b_i^n = |\ \Delta_i^n X\ |^r |\ \Delta_{i+1}^n X\ |^s$ but suppress the dependence of b_i^n on r, s in the notation for sake of a simple notation. The i.i.d. bootstrap analogue of b_i^n is labeled as b_i^{n*}. The data b_i^{n*} are calculated as

$$b_i^{n*} = b_{I_i}^n, \qquad I_i \sim \text{i.i.d. uniformly distributed on } \{1, \cdots, n\}. \tag{17}$$

The resulting bootstrap statistic is then defined as

$$V^*(X,r,s)^n = n^{\frac{r+s}{2}-1} \sum_{i=1}^n b_i^{n*}. \tag{18}$$

The mean value of this bootstrap statistic is given by[2]

$$E^*(V^*(X,r,s)^n) = E^*\left(n^{\frac{r+s}{2}-1} \sum_{i=1}^n b_i^{n*}\right) = n^{\frac{r+s}{2}} E^*(b_1^{n*}) = V(X,r,s)^n. \tag{19}$$

[2] E^* denotes expectation with respect to the i.i.d. bootstrap data, conditional on the original sample.

Equation (17) can be easily verified since b_i^{n*} is an i.i.d. sample from $(b_i^n), i = 1, ..., n$. Moreover, we obtain the identity[3]

$$\rho^{*2}(r,s) = Var^*(\sqrt{n}V^*(X,r,s)^n) = nVar^*\left(n^{\frac{r+s}{2}-1}\sum_{i=1}^{n} b_i^{n*}\right)$$

$$= n^{r+s}Var^*(b_1^{n*}) = V(X,2r,2s)^n - [V(X,r,s)^n]^2. \tag{20}$$

We use

$$\rho^{*2}(r,s)^n = V^*(X,2r,2s)^n - [V^*(X,r,s)^n]^2 \tag{21}$$

as a consistent estimator for $\rho^{*2}(r,s)$.
The i.i.d. bootstrap analogues of T_n and S_n are given by

$$T_n^* = \frac{\sqrt{n}(V^*(X,r,s)^n - V(X,r,s)^n)}{\rho^*(r,s)^n} \tag{22}$$

and

$$S_n^* = \frac{\sqrt{n}(V^*(X,r,s)^n - V(X,r,s)^n)}{\rho^*(r,s)}, \tag{23}$$

respectively.

If we use this notation and assume that $X_t \in BSM$, we can formulate the following theorem.

Theorem 1
For $n \to \infty$ we have

$$\sup_{x\in\mathbf{R}} \mid P^*\left(T_n^* \le x\right) - \Phi(x) \mid \xrightarrow{\mathcal{P}} 0, \tag{24}$$

where $\Phi(x) = P(Z \le x)$ with $Z \sim N(0,1)$.

If assumption ($\mathbf{V}$) holds, this implies for $n \to \infty$:

$$P^*(T_n^* \le x) - P(T_n \le x) = o_P(1) \tag{25}$$

uniformly in $x \in \mathbf{R}$.

Proof
The proof is divided in two steps. First, we show that the result is true for S_n^*. Then it remains to show $\rho^*(r,s)^n \xrightarrow{\mathcal{P}^*} \rho^*(r,s)$.

1. Step

Define

$$z_i^* = \frac{n^{\frac{r+s-1}{2}}\left(b_i^{n*} - E^*(b_i^{n*})\right)}{\rho^*(r,s)} \tag{26}$$

and

$$s_n^* = \sum_{i=1}^{n} z_i^*. \tag{27}$$

[3] Var^* denotes variance with respect to the i.i.d. bootstrap data, conditional on the original sample.

Here the z_i^* are (conditioned on the original sample) i.i.d. and we have

$$E^*(z_i^*) = 0. \tag{28}$$

Furthermore, we obtain

$$Var^*(z_i^*) = \frac{n^{r+s-1}}{\rho^{*2}(r,s)} Var(b_i^{n*}) = \frac{1}{n}. \tag{29}$$

The last equation yields

$$Var^*\left(\sum_{i=1}^n z_i^*\right) = 1. \tag{30}$$

Now we are in a position to use Berry-Esseen's bound (see Katz (1963) pages 1107-1108 for more details). For some small $\epsilon > 0$ and some constant $K > 0$ it holds that

$$\sup_{x\in\mathbf{R}} \mid P^*\left(\frac{\sum_{i=1}^n z_i^*}{\sqrt{Var^*(\sum_{i=1}^n z_i^*)}} \le x\right) - \Phi(x) \mid \le K \sum_{i=1}^n E^* \mid z_i^* \mid^{2+\epsilon}. \tag{31}$$

Now we have to show, that the right hand side converges to zero in probability. We get

$$\begin{aligned}
&\sum_{i=1}^n E^* \mid z_i^* \mid^{2+\epsilon} = nE^* \mid z_1^* \mid^{2+\epsilon} \\
&= n^{\frac{(r+s-2)(2+\epsilon)+4+\epsilon}{2}} \mid \rho^*(r,s)_t \mid^{\frac{-2+\epsilon}{2}} E^*(\mid b_i^{n*} - E^*(b_i^{n*}) \mid^{2+\epsilon}) \\
&\le 2 \mid \rho^*(r,s) \mid^{\frac{-2+\epsilon}{2}} n^{\frac{2r+2s+r\epsilon+s\epsilon-\epsilon}{2}} E^*(b_i^{n*})^{2+\epsilon} \\
&= 2 \mid \rho^*(r,s) \mid^{\frac{-2+\epsilon}{2}} n^{\frac{-\epsilon}{2}} V(X, r(2+\epsilon), s(2+\epsilon))_t^n = O_p(n^{\frac{-\epsilon}{2}}).
\end{aligned} \tag{32}$$

This is sufficient for the convergence as we already know that

$$V(X,r,s)^n \xrightarrow{\mathcal{P}} V(X,r,s) = \mu_r\mu_s \int_o^1 \mid \sigma_u \mid^{r+s} du \tag{33}$$

and

$$\rho^*(r,s) \xrightarrow{\mathcal{P}} \mu_{2r}\mu_{2s} \int_0^1 \mid \sigma_u \mid^{2r+2s} du - \mu_r^2\mu_s^2 \left(\int_0^1 \mid \sigma_u \mid^{r+s} du\right)^2 > 0. \tag{34}$$

□

2. Step

We are left to proving that

$$\rho^*(r,s)^n \xrightarrow{\mathcal{P}^*} \rho^*(r,s). \tag{35}$$

For this purpose we prove that

$$Bias(\rho^*(r,s)^n) = E^*(\rho^*(r,s)^n) - \rho^*(r,s) \xrightarrow{\mathcal{P}} 0 \tag{36}$$

and

$$Var^*(\rho^*(r,s)^n) \xrightarrow{\mathcal{P}} 0. \tag{37}$$

This follows from Lemma 4 (part i) and j)) and completes the proof. □

4 Wild bootstrap

The wild bootstrap is based on a different principle then the i.i.d. bootstrap. It uses the same summands as the original statistic $V(X,r,s)^n$, but the returns are all multiplied by an external random variable. More precisely, we define

$$V^{WB}(X,r,s)^n = n^{\frac{r+s}{2}-1} \sum_{i=1}^{n} \eta_i^2 \mid \Delta_i^n X \mid^r \mid \Delta_{i+1}^n X \mid^s . \tag{38}$$

The (η_i) are i.i.d. external random variables, which are independent of the original process. We use the notation

$$\mu_m^{WB} = E \mid \eta_i \mid^m \tag{39}$$

to denote the absolute moments of η_i.
It can be easily seen that[4]

$$E^{WB}(V^{WB}(X,r,s)^n) = \mu_2^{WB} V(X,r,s)^n. \tag{40}$$

The variance of $\sqrt{n}V^{WB}(X,r,s)^n$ is given by[5]

$$\begin{aligned}
\rho_{WB}^2(r,s) &= Var^{WB}(\sqrt{n}V^{WB}(X,r,s)^n) \\
&= n[E^{WB}\left((V^{WB}(X,r,s)^n)^2\right) - \left(E^{WB}(V^{WB}(X,r,s)^n)\right)^2] \\
&= n^{r+s-1}\left(\mu_4^{WB} - (\mu_2^{WB})^2\right) \sum_{i=1}^{n} \mid \Delta_i^n X \mid^{2r} \mid \Delta_{i+1}^n X \mid^{2s} \\
&= \left(\mu_4^{WB} - (\mu_2^{WB})^2\right) V(X,2r,2s)^n.
\end{aligned} \tag{41}$$

We use the following consistent estimator for the variance term
$\rho_{WB}^2(r,s) = \left(\mu_4^{WB} - (\mu_2^{WB})^2\right) V(X,2r,2s)^n$:

$$\rho_{WB}^2(r,s)^n = \frac{\mu_4^{WB} - (\mu_2^{WB})^2}{\mu_4^{WB}} n^{\frac{2r+2s}{2}-1} \sum_{i=1}^{n} \eta_i^4 \mid \Delta_i^n X \mid^{2r} \mid \Delta_{i+1}^n X \mid^{2s} . \tag{42}$$

With these notations we can formulate the following theorem.

Theorem 2
Assume that $\mu_q^{WB} < \infty$ for $q = 2(2+\epsilon)$ and define

$$T_n^{WB} = \frac{\sqrt{n}(V^{WB}(X,r,s)^n - \mu_2^{WB} V(X,r,s)^n)}{\rho_{WB}(r,s)^n} \tag{43}$$

and

$$S_n^{WB} = \frac{\sqrt{n}(V^{WB}(X,r,s)^n - \mu_2^{WB} V(X,r,s)^n)}{\rho_{WB}(r,s)}. \tag{44}$$

Then for $n \to \infty$ the following holds:

$$\sup_{x \in \mathbf{R}} \mid P^{WB}(T_n^{WB} \le x) - \Phi(x) \mid \xrightarrow{\mathcal{P}} 0. \tag{45}$$

[4] E^{WB} denotes expectation with respect to the wild bootstrap data, conditional on the original sample.
[5] Var^{WB} denotes variance with respect to the wild bootstrap data, conditional on the original sample.

Obviously, this is a wild bootstrap analogue of Theorem 1.

Proof
The proof is similar to that of Theorem 1. First, we show that the statement is true for S_n^{WB}. After that, it remains to show that the variance converges in the right sense.
First, we define

$$x_i = \frac{n^{\frac{r+s-1}{2}}(\eta_i^2 - \mu_2^{WB}) \mid \Delta_i^n X \mid^r \mid \Delta_{i+1}^n X \mid^s}{\rho_{WB}(r,s)}. \tag{46}$$

This yields, of course, $E^{WB}(x_i) = 0$. Moreover, we obtain

$$Var^{WB}(x_i) = \frac{n^{r+s-1}(\mu_4^{WB} - (\mu_2^{WB})^2) \mid \Delta_i^n X \mid^{2r} \mid \Delta_{i+1}^n X \mid^{2s}}{\rho_{WB}^2(r,s)} \tag{47}$$

and therefore

$$Var^{WB}\left(\sum_{i=1}^n x_i\right) = \sum_{i=1}^n Var^{WB}(x_i) = 1. \tag{48}$$

Now we use again Berry-Esseen's bound. The rest of this step is analogue to the proof of Theorem 1 and yields:

$$\sum_{i=1}^n E^{WB} \mid x_i^* \mid^{2+\epsilon} = O_p(n^{\frac{-\epsilon}{2}}). \tag{49}$$

□

In the second step we have on the one hand

$$E^{WB}(\rho_{WB}^2(r,s)^n) = \frac{\mu_4^{WB} - (\mu_2^{WB})^2}{\mu_4^{WB}} n^{\frac{2r+2s}{2}-1} \sum_{i=1}^n E^{WB}(\eta_i^4) \mid \Delta_i^n X \mid^{2r} \mid \Delta_{i+1}^n X \mid^{2s} = \rho_{WB}^2(r,s). \tag{50}$$

On the other hand we obtain for the variance

$$Var^{WB}(\rho_{WB}^2(r,s)^n) = E^{WB}[(\rho_{WB}^2(r,s)^n)^2] - E^{WB}(\rho_{WB}^2(r,s))^2. \tag{51}$$

Calculating the mean values yields

$$\begin{aligned} E^{WB}[(\rho_{WB}^2(r,s)^n)^2] = &\left(\frac{\mu_4^{WB} - (\mu_2^{WB})^2}{\mu_4^{WB}}\right)^2 n^{2r+2s-2} E^{WB}\Big(\sum_{i=1}^n \eta_i^8 \mid \Delta_i^n X \mid^{4r} \mid \Delta_{i+1}^n X \mid^{4s} \\ &+ \sum_{i=1}^n \eta_i^4 \eta_{i+1}^4 \mid \Delta_i^n X \mid^{2r} \mid \Delta_{i+1}^n X \mid^{2r+2s} \mid \Delta_{i+2}^n X \mid^{2s} \\ &+ \sum_{i=1}^n \eta_i^4 \eta_{i-1}^4 \mid \Delta_{i-1}^n X \mid^{2r} \mid \Delta_i^n X \mid^{2r+2s} \mid \Delta_{i+1}^n X \mid^{2s} \\ &+ \sum_{i=1}^n \sum_{j=1, j\neq i,i+1,i-1}^n \eta_i^4 \eta_j^4 \mid \Delta_i^n X \mid^{2r} \mid \Delta_{i+1}^n X \mid^{2s} \mid \Delta_j^n X \mid^{2r} \mid \Delta_{j+1}^n X \mid^{2s}\Big) \end{aligned} \tag{52}$$

and

$$E^{WB}(\rho_{WB}^2(r,s))^2 = (\mu_4^{WB} - (\mu_2^{WB})^2)^2 n^{2r+2s-2}\Big(\sum_{i=1}^{n} \mid \Delta_i^n X \mid^{4r} \mid \Delta_{i+1}^n X \mid^{4s}$$
$$+\sum_{i=1}^{n} \mid \Delta_i^n X \mid^{2r} \mid \Delta_{i+1}^n X \mid^{2r+2s} \mid \Delta_{i+2}^n X \mid^{2s}$$
$$+\sum_{i=1}^{n} \mid \Delta_{i-1}^n X \mid^{2r} \mid \Delta_i^n X \mid^{2r+2s} \mid \Delta_{i+1}^n X \mid^{2s}$$
$$+\sum_{i=1}^{n} \sum_{j=1, j\neq i,i+1,i-1}^{n} \mid \Delta_i^n X \mid^{2r} \mid \Delta_{i+1}^n X \mid^{2s} \mid \Delta_j^n X \mid^{2r} \mid \Delta_{j+1}^n X \mid^{2s}\Big). \tag{53}$$

Calculating the difference

$$E^{WB}[(\rho_{WB}^2(r,s)^n)^2] - E^{WB}(\rho_{WB}^2(r,s))^2$$
$$= \left(\frac{\mu_4^{WB} - (\mu_2^{WB})^2}{\mu_4^{WB}}\right)^2 \mu_8^{WB} n^{2r+2s-2} \sum_{i=1}^{n} \mid \Delta_i^n X \mid^{4r} \mid \Delta_{i+1}^n X \mid^{4s}$$
$$-(\mu_4^{WB} - (\mu_2^{WB})^2)^2 n^{2r+2s-2} \sum_{i=1}^{n} \mid \Delta_i^n X \mid^{4r} \mid \Delta_{i+1}^n X \mid^{4s} = O\left(\frac{1}{n}\right) \tag{54}$$

completes the proof. □

5 Second order accuracy of the bootstrap

In the last two sections we have shown that both bootstrap methods are first order accurate. Now we present some arguments for the improvements, which we see later in the simulations. We will find there, that confidence sets obtained with the bootstrap methods are much more accurate then the ones of the normal distribution (see Gonçalves and Meddahi (2004) or section 6).

For this reason we study the second-order Edgeworth expansion for the distribution of T_n (see, e.g., Hall (1992) page 48)

$$P(T_n \leq x) = \Phi(x) + n^{-\frac{1}{2}} q_1 \phi(x) + O\left(\frac{1}{n}\right). \tag{55}$$

The quantity q_1 is given by

$$q_1 = -k_1 - \frac{1}{6} k_3 (x^2 - 1), \tag{56}$$

where

$$k_1 = E(T_n) \tag{57}$$

and

$$k_3 = E(T_n^3) - 3E(T_n^2)E(T_n) + 2[E(T_n)]^3. \tag{58}$$

Remark 2

i) Formal proofs of the Edgeworth expansions are only possible under very strong assumptions.

But it is usual to use this formal expression for investigations concerning higher order accuracy (see, e.g., Mammen (1993) or Davidson and Flachaire (2001)).
ii) For the distributions of T_n^* and T_n^{WB} we define the Edgeworth expansions in an analogue way.

In the calculation of the cumulants we use the following notation

$$U_n = \frac{\sqrt{n}(\rho^2(r,s)^n - \rho^2(r,s))}{\rho^2(r,s)}. \tag{59}$$

This yields

$$T_n = S_n \left(\frac{\rho^2(r,s)^n}{\rho^2(r,s)} \right)^{-\frac{1}{2}} = S_n \left(1 + \frac{U_n}{\sqrt{n}} \right)^{-\frac{1}{2}}. \tag{60}$$

By Taylor expansion we obtain for $f(x) = (1+x)^{-\frac{k}{2}}$

$$f(x) = 1 - \frac{k}{2}x + O(x^2). \tag{61}$$

Having this in mind we can rewrite T_n^k as

$$T_n^k = S_n^k \left(1 + \frac{U_n}{\sqrt{n}} \right)^{-\frac{k}{2}} = S_n^k - \frac{k}{2}\frac{S_n^k U_n}{\sqrt{n}} + O\left(\frac{1}{n}\right) = \tilde{T}_n^k + O\left(\frac{1}{n}\right). \tag{62}$$

To derive the forthcoming results we need the rather strong assumptions that $\sigma_s = \sigma \geq 0$ is constant and $b_s = 0$. This yields the simplified model

$$X_t = X_0 + \int_0^t \sigma dW_s,\ t \in [0,1]. \tag{63}$$

Since the drift process b is of small order (compared to the volatility σ) when data are observed at high frequencies (see, e.g., Andersen et al. (2002)) this restriction is not too crucial.

Remark 3
If $\sigma_s = \sigma$ is constant, we obtain

$$\Delta_i^n X = \int_{\frac{i-1}{n}}^{\frac{i}{n}} \sigma_u dW_u = \sigma(W_{\frac{i}{n}} - W_{\frac{i-1}{n}}) = \sigma v_i. \tag{64}$$

So the $\Delta_i^n X$ are i.i.d. with common distribution $N(0, \frac{\sigma^2}{n})$, whereas $v_i = W_{\frac{i}{n}} - W_{\frac{i-1}{n}} \sim N(0, \frac{1}{n})$.

With this additional assumption we can show the following estimate for the i.i.d. bootstrap in the simplified model

$$plim_{n\to\infty} \mid q_1^*(x) - q_1(x) \mid \leq \mid q_1(x) \mid. \tag{65}$$

This suggests that the bootstrap error is asymptotically smaller than one made by normal approximation.
For the wild bootstrap we can, if the distribution of η_i is appropriately chosen, even show that

$$plim_{n\to\infty} \mid q_1^{WB}(x) - q_1(x) \mid = 0. \tag{66}$$

This means we have a second order refinement.

Remark 4
i) For the i.i.d. bootstrap we have only $plim_{n\to\infty} \mid q_1^*(x) - q_1(x) \mid = 0$, if r=2 and s=0 (see Gonçalves and Meddahi (2004) and Tables 17 and 18).
ii) Even if σ is not constant, simulation results show that both bootstrap distributions match the distribution of T_n better than the normal distribution (see Gonçalves and Meddahi (2004) or Section 6).

Lemma 1
Under the additional assumption that σ is constant, we obtain in the simplified model (63) the following results for $n \to \infty$

$$\text{a) } k_1 = \frac{-A_1}{2\sqrt{n}\rho^3(r,s)}(\mu_{3r}\mu_{3s} + \mu_{2r}\mu_s\mu_{r+2s} + \mu_r\mu_{2s}\mu_{2r+s} - 3\mu_r\mu_s\mu_{2r}\mu_{2s})\sigma^{3(r+s)} + O\left(\frac{1}{n}\right) \quad (67)$$

and

$$A_1 = \frac{\mu_{2r}\mu_{2s} + 2\mu_r\mu_s\mu_{r+s} - 3\mu_r^2\mu_s^2}{\mu_{2r}\mu_{2s}} \quad (68)$$

$$\begin{aligned} \text{b) } k_3 &= \frac{\sigma^{3(r+s)}}{2\sqrt{n}\rho^3(r,s)\mu_{2r}\mu_{2s}}(-14\mu_r^3\mu_s^3\mu_{2r}\mu_{2s} - 12\mu_{3r}\mu_{3s}\mu_r\mu_s\mu_{r+s} - 12\mu_r\mu_s^2\mu_{r+s}\mu_{2r}\mu_{r+2s} \\ &- 12\mu_r^2\mu_s\mu_{r+s}\mu_{2s}\mu_{2r+s} - 12\mu_r^2\mu_s^2\mu_{r+s}\mu_{2r}\mu_{2s} + 18\mu_{3r}\mu_{3s}\mu_r^2\mu_s^2 + 18\mu_r^2\mu_s^3\mu_{2r}\mu_{r+2s} \\ &+ 18\mu_r^3\mu_s^2\mu_{2s}\mu_{2r+s} + 12\mu_{r+s}^2\mu_{2r}\mu_{2s}\mu_r\mu_s - 4\mu_{3r}\mu_{3s}\mu_{2r}\mu_{2s}) + O\left(\frac{1}{n}\right) \end{aligned} \quad (69)$$

$$\text{c) } k_1^* = \frac{-1}{2\sqrt{n}\rho^{*3}(r,s)}(V(X,3r,3s)^n - 3V(X,r,s)^nV(X,2r,2r)^n + 2(V(X,r,s)^n)^3) + O\left(\frac{1}{n}\right) \quad (70)$$

$$\text{d) } k_3^* = \frac{-2}{\sqrt{n}\rho^{*3}(r,s)}(V(X,3r,3s)^n - 3V(X,r,s)^nV(X,2r,2r)^n + 2(V(X,r,s)^n)^3) + O\left(\frac{1}{n}\right) \quad (71)$$

$$\text{e) } k_1^{WB} = E^{WB}(T_n^{WB}) = \frac{-1}{2\sqrt{n}}\left(\frac{\mu_6^{WB} - \mu_2^{WB}\mu_4^{WB}}{(\mu_4^{WB} - (\mu_2^{WB})^2)^{\frac{1}{2}}\mu_4^{WB}}\right)\frac{V(X,3r,3s)^n}{(V(X,2r,2s)^n)^{\frac{3}{2}}} + O\left(\frac{1}{n}\right) \quad (72)$$

$$\begin{aligned} \text{f) } k_3^{WB} &= \frac{V(X,3r,3s)^n}{\sqrt{n}(V(X,2r,2s)^n)^{\frac{3}{2}}}\left(\frac{\mu_6^{WB} - 3\mu_4^{WB}\mu_2^{WB} + 2(\mu_2^{WB})^3}{(\mu_4^{WB} - (\mu_2^{WB})^2)^{\frac{3}{2}}} - \frac{3(\mu_6^{WB} - \mu_4^{WB}\mu_2^{WB})}{\mu_4^{WB}(\mu_4^{WB} - (\mu_2^{WB})^2)^{\frac{1}{2}}}\right) \\ &+ O\left(\frac{1}{n}\right) \end{aligned} \quad (73)$$

Proof
The values for $E(S_n^1)$, $E(S_n^3)$, $E(S_n^1 U_n)$ and $E(S_n^3 U_n)$ can be found in Lemma 3. These values are derived using Lemma 2 for the equations

$$E(S_n^1) = n^{\frac{1}{2}} E\left(\frac{V(X,r,s)^n - V(X,r,s)}{\rho(r,s)}\right), \tag{74}$$

$$E(S_n^3) = n^{\frac{3}{2}} E\left(\frac{(V(X,r,s)^n - V(X,r,s))^3}{\rho^3(r,s)}\right), \tag{75}$$

$$E(S_n^1 U_n) = nE\left(\frac{(V(X,r,s)^n - V(X,r,s))(\rho^2(r,s)^n - \rho 2(r,s))}{\rho^3(r,s)}\right) \tag{76}$$

and

$$E(S_n^3 U_n) = n^2 E\left(\frac{(V(X,r,s)^n - V(X,r,s))^3(\rho^2(r,s)^n - \rho 2(r,s))}{\rho^5(r,s)}\right). \tag{77}$$

The corresponding results for the i.i.d. bootstrap are listed in Lemma 4 and Lemma 5, respectively in Lemma 6 for the wild bootstrap. Combining this yields the desired results. □

If we compare the limits of q_1^{WB} and $q_1(x)$ and want to choose the distribution of η_i such that the difference becomes 0, we get a system of 2 equations and 3 variables, which has infinitely many solutions. Namely, we have to solve

$$k_1 = k_1^{WB} \tag{78}$$

and

$$k_3 = k_3^{WB}, \tag{79}$$

in which μ_2^{WB}, μ_4^{WB} and μ_6^{WB} are the variables.
If we choose a distribution for η_i in such a way that the system of equations is solved, we find a distribution which yields a second order refinement. For r=2 and s=0 there is an example for a correct specification in Gonçalves and Meddahi (2004). For r=1, s=1 we propose for example the distribution $\eta_i = 0,828838488$ with probability $p = 0,778271187$ and $\eta_i = 1,44869881$ with probability $1-p$. For r=4, s=0 the distribution $\eta_i = 1,353047$ with probability $p = 0,268306$ and $\eta_i = 0,833891$ with probability $1-p$ yields a second order refinement.
As a consequence we obtain the following results under the same assumptions as in Lemma 1.

Corollary 1

$$plim_{n\to\infty} \mid q_1^*(x) - q_1(x) \mid \leq \mid q_1(x) \mid . \tag{80}$$

Corollary 2
If the distribution of the η_i is chosen as above, it holds that

$$plim_{n\to\infty} \mid q_1^{WB}(x) - q_1(x) \mid = 0. \tag{81}$$

Remark 5
i) The proofs of these results are not too difficult but very lengthy and tedious. They follow directly from the technical results in the Appendix and Lemma 1. For this reason the proofs are omitted here.

ii) Tables 17 and 18 in the Appendix contain values for k_1, $\mid k_1 - k_1^* \mid$, k_3, and $\mid k_3 - k_3^* \mid$ as given in Lemma 1 for some values of r and s. This illustrates the statement of Corollary 1.

6 Confidence sets

In the last section we have shown that both bootstrap methods, under quite strong assumptions, approximate the distribution of T_n better than the standard normal distribution. An important application of this result is the construction of confidence sets for $V(X, r, s)$.
A 2-sided $100(1-\alpha)\%$ confidence interval is given by

$$C_2 = \left(V(X,r,s)^n - z_{\frac{1-\alpha}{2}} \sqrt{\frac{\rho^2(r,s)^n}{n}}, V(X,r,s)^n + z_{\frac{1-\alpha}{2}} \sqrt{\frac{\rho^2(r,s)^n}{n}} \right). \tag{82}$$

A 1-sided $100(1-\alpha)\%$ confidence interval is obtained by

$$C_1 = \left(0, V(X,r,s)^n - z_{\alpha} \sqrt{\frac{\rho^2(r,s)^n}{n}} \right). \tag{83}$$

Here is z_α the critical value of the standard normal distribution for an α-level.
Using the introduced bootstrap methods we obtain some different confidence sets

$$C_2^{Boot} = \left(V(X,r,s)^n - z_{\frac{1-\alpha}{2}}^{Boot} \sqrt{\frac{\rho^2(r,s)^n}{n}}, V(X,r,s)^n - z_{\frac{\alpha}{2}}^{Boot} \sqrt{\frac{\rho^2(r,s)^n}{n}} \right) \tag{84}$$

and

$$C_1^{Boot} = \left(0, V(X,r,s)^n - z_{\alpha}^{Boot} \sqrt{\frac{\rho^2(r,s)^n}{n}} \right). \tag{85}$$

Here z_α^{Boot} is the α-quantile of the bootstrap distribution T_n^{Boot}, that is $T_n^{Boot} = T_n^*$ or $T_n^{Boot} = T_n^{WB}$. Simulation results for r=2 and s=0 can be found in Gonçalves and Meddahi (2004).
In this paper we present some results for r=4, s=0 and r=1, s=1. As already mentioned, these two statistics have a lot of applications in practice. 1,1-bipower is robust against jumps. Moreover, it can be used to construct tests for the presence of jumps in asset prices (see BN-S (2006) and Christensen and Podolskij (2006a)). 4,0-bipower is used in central limit theorems concerning integrated volatility to estimate the variance (BN-S (2004b)).

Remark 6
i) BN-S (2002) introduced a log version of the confidence sets C_1 and C_2. Of course, it is possible to construct analogue confidence sets for C_1^{Boot} and C_2^{Boot} too.
ii) It is possible to extend Theorem 1 and Theorem 2 for range-based estimators (see, e.g., Christensen und Podolskij (2006a, b)).

6.1 Finite sample results

We investigate the behaviour of the bootstrap methods for r=4, s=0 and r=1, s=1 (see Tables 1-16). We calculate 1- and 2-sided 95%-confidence sets for constant volatility and a 2-factor-model. The latter is specified by

$$dX_t = \mu dt + \sigma_t dW_t, \tag{86}$$

$$\sigma_t = exp(\beta_0 + \beta_1 \tau_t), \tag{87}$$

$$d\tau_t = \alpha \tau_t dt + dB_t, \tag{88}$$

$$corr(dW_t, dB_t) = \rho. \tag{89}$$

The parameters are chosen as $\beta_0 = 0,3125$, $\beta_1 = 0,125$, $\alpha = -0,025$ and $\rho = -0,3$. Moreover, we study the influence of the drift function. For this purpose we choose $\mu = 0$ in the first run and $\mu = 0,03$ in the second. The 2-factor-model can also be found in Barndorff-Nielsen, Hansen, Lunde and Shephard (2006) and Podolskij and Vetter (2006). We did 1000 simulation runs with 600 bootstrap replications each. For the wild bootstrap we used the distributions proposed in Section 5.

The results are listed in detail in Tables 1-16 in the Appendix. We use the abbreviation *Nor* for the level obtained by the normal distribution. Moreover, *i.i.d. Boot* and *WB* are the abbreviations for the level obtained by the i.i.d. bootstrap distribution and the wild bootstrap distribution, respectively. The ending *-log* symbolizes the level obtained by the log version of the corresponding distribution

In all cases the intervals tend to undercover. The degree of undercoverage is especially large for small sample sizes. But it is obvious that the bootstrap intervals are much more accurate then the ones obtained by the normal distribution. Especially for small sample sizes a significant improvement can be observed. Furthermore, the log-versions improve the accuracy additionally. Besides that, the 2-sided confidence sets seem to be more accurate than the 1-sided. Moreover, the simulations show that the results are robust to drift effects.

For r=4, s=0 the i.i.d. bootstrap yields the best results, followed by the wild bootstrap. For the case r=1, s=1 it is the same situation if there are only a few data points. But if the sample size grows the results become more and more similar.

7 Appendix

For the proofs in section 5 we have to calculate a lot of different moments. This results in tedious, but simple, calculations. The principles of these calculations can be found in Gonçalves and Meddahi (2004). For this reason the proofs are omitted here.
In the simplified model and under the same assumptions as in Lemma 1 we obtain the following results.

Lemma 2

a)

$$E(V(X,r,s)^n) = V(X,r,s) \tag{90}$$

b)

$$E([V(X,r,s)^n]^2) = (V(X,r,s))^2 + \frac{\rho^2(r,s)}{n} \tag{91}$$

c)

$$\begin{aligned} E([V(X,r,s)^n]^3) &= \sigma^{3(r+s)}[\mu_s^3\mu_r^3 + \frac{3\mu_{2r}\mu_{2s}\mu_r\mu_s + 6\mu_{r+s}\mu_r^2\mu_s^2 - 9\mu_s^3\mu_r^3}{n} \\ &+ \frac{\mu_{3r}\mu_{3s} + 3\mu_{2r+s}\mu_r\mu_{2s} + 3\mu_{2s+r}\mu_{2r}\mu_s - 9\mu_{2r}\mu_r\mu_s\mu_{2s} + 6\mu_{r+s}^2\mu_r\mu_s}{n^2} \\ &+ \frac{-24\mu_{r+s}\mu_r^2\mu_s^2 + 20\mu_r^3\mu_s^3}{n^2}] \end{aligned} \tag{92}$$

d)

$$E(\rho^2(r,s)^n) = \rho^2(r,s) \tag{93}$$

e)

$$\begin{aligned} &E(V(X,r,s)^n\rho^2(r,s)^n) = V(X,r,s)\rho^2(r,s) \\ &+ A_1\sigma^{3(r+s)}\frac{\mu_{3r}\mu_{3s} + \mu_{2r}\mu_s\mu_{r+2s} + \mu_r\mu_{2s}\mu_{2r+s} - 3\mu_r\mu_s\mu_{2r}\mu_{2s}}{n} \end{aligned} \tag{94}$$

f)

$$E([V(X,r,s)^n]^2\rho^2(r,s)^n) = \sigma^{4(r+s)}A_1[\mu_r^2\mu_s^2\mu_{2r}\mu_{2s} + \frac{\mu_{2r}^2\mu_{2s}^2 + 2\mu_{r+s}\mu_r\mu_s\mu_{2r}\mu_{2s}}{n}$$
$$+\frac{2\mu_{3r}\mu_{3s}\mu_r\mu_s + 2\mu_{r+2s}\mu_{2r}\mu_r\mu_s^2 + 2\mu_{2r+s}\mu_{2s}\mu_r^2\mu_s - 9\mu_r^2\mu_s^2\mu_{2r}\mu_{2s}}{n}$$
$$+\frac{\mu_{4r}\mu_{4s} + 2\mu_{2r+2s}\mu_{2r}\mu_{2s} - 3\mu_{2r}^2\mu_{2s}^2 + 2\mu_{3r+s}\mu_{3s}\mu_r + 2\mu_{r+3s}\mu_{3r}\mu_s}{n^2}$$
$$+\frac{2\mu_{r+s}\mu_{2r+s}\mu_r\mu_{2s} + 2\mu_{r+s}\mu_{r+2s}\mu_{2r}\mu_s + 2\mu_{2r+s}\mu_{r+2s}\mu_r\mu_s - 6\mu_{3r}\mu_{3s}\mu_r\mu_s}{n^2}$$
$$\frac{-8\mu_r\mu_s^2\mu_{2r}\mu_{r+2s} - 8\mu_r^2\mu_s\mu_{2r+s}\mu_{2s} - 8\mu_{r+s}\mu_s\mu_r\mu_{2r}\mu_{2s} + 20\mu_{2r}\mu_{2s}\mu_r^2\mu_s^2}{n^2}] \tag{95}$$

g)

$$E([V(X,r,s)^n]^3\rho^2(r,s)^n) = \sigma^{5(r+s)}A_1[\mu_r^3\mu_s^3\mu_{2r}\mu_{2s} + \frac{-18\mu_r^3\mu_s^3\mu_{2r}\mu_{2s}}{n}$$
$$+\frac{3\mu_r^3\mu_s^2\mu_{2r+s}\mu_{2s} + 3\mu_r^2\mu_s^3\mu_{r+2s}\mu_{2r} + 6\mu_r^2\mu_s^2\mu_{2r}\mu_{2s}\mu_{r+s} + 3\mu_{2r}^2\mu_{2s}^2\mu_r\mu_s}{n}$$
$$+\frac{3\mu_r^2\mu_s^2\mu_{3r}\mu_{3s}}{n} + \frac{107\mu_r^3\mu_s^3\mu_{2r}\mu_{2s} - 33\mu_r^3\mu_s^2\mu_{2r+s}\mu_{2s} - 33\mu_r^2\mu_s^3\mu_{r+2s}\mu_{2r}}{n^2}$$
$$+\frac{-66\mu_r^2\mu_s^2\mu_{2r}\mu_{2s}\mu_{r+s} + 6\mu_r^2\mu_{r+s}\mu_s\mu_{2s}\mu_{2r+s} + 6\mu_s^2\mu_{r+s}\mu_r\mu_{2r}\mu_{r+2s} - 27\mu_{2r}^2\mu_{2s}^2\mu_r\mu_s}{n^2}$$
$$+\frac{-27\mu_r^2\mu_s^2\mu_{3r}\mu_{3s} + 6\mu_{r+s}^2\mu_r\mu_s\mu_{2r}\mu_{2s} + 6\mu_{r+s}\mu_{r+2s}\mu_r\mu_s^2\mu_{2r} + 6\mu_{r+2s}\mu_{2r+s}\mu_r^2\mu_s^2}{n^2}$$
$$+\frac{6\mu_{2r+s}\mu_{r+s}\mu_r^2\mu_s\mu_{2s} + 4\mu_{3r}\mu_{3s}\mu_{2r}\mu_{2s} + 3\mu_{4r}\mu_{4s}\mu_r\mu_s + 3\mu_{r+2s}\mu_{2r}^2\mu_{2s}\mu_s}{n^2}$$
$$+\frac{3\mu_{2r+s}\mu_{2r}\mu_r\mu_{2s}^2 + 6\mu_{r+s}\mu_{3r}\mu_{3s}\mu_r\mu_s + 3\mu_{2r+s}\mu_{2r}\mu_r\mu_{2s}^2 + 3\mu_{r+2s}\mu_{2r}^2\mu_{2s}\mu_s}{n^2}$$
$$+\frac{6\mu_{2r+2s}\mu_{2r}\mu_r\mu_{2s}\mu_s + 6\mu_{3r+s}\mu_r^2\mu_{3s}\mu_s + 6\mu_{r+3s}\mu_{3r}\mu_r\mu_s^2}{n^2}$$
$$+\frac{-210\mu_r^3\mu_s^3\mu_{2r}\mu_{2s} + 90\mu_r^3\mu_s^2\mu_{2r+s}\mu_{2s} + 90\mu_r^2\mu_s^3\mu_{r+2s}\mu_{2r} + 180\mu_r^2\mu_s^2\mu_{2r}\mu_{2s}\mu_{r+s}}{n^3}$$
$$+\frac{-30\mu_r^2\mu_{r+s}\mu_s\mu_{2s}\mu_{2r+s} - 30\mu_s^2\mu_{r+s}\mu_r\mu_{2r}\mu_{r+2s} + 60\mu_{2r}^2\mu_{2s}^2\mu_r\mu_s + 60\mu_r^2\mu_s^2\mu_{3r}\mu_{3s}}{n^3}$$
$$+\frac{6\mu_{r+s}^2\mu_{2r}\mu_s\mu_{r+2s} + 12\mu_{r+s}\mu_r\mu_s\mu_{r+2s}\mu_{2r+s} + 6\mu_{r+s}^2\mu_r\mu_{2s}\mu_{2r+s} - 30\mu_{r+s}^2\mu_r\mu_s\mu_{2r}\mu_{2s}}{n^3}$$

$$+\frac{-30\mu_{r+s}\mu_{r+2s}\mu_r\mu_s^2\mu_{2r} - 30\mu_{r+2s}\mu_{2r+s}\mu_r^2\mu_s^2 - 30\mu_{2r+s}\mu_{r+s}\mu_r^2\mu_s\mu_{2s} + \mu_{5r}\mu_{5s}}{n^3}$$

$$+\frac{4\mu_{3r+2s}\mu_{2r}\mu_{3s} + 4\mu_{2r+3s}\mu_{3r}\mu_{2s} - 12\mu_{3r}\mu_{3s}\mu_{2r}\mu_{2s} + 3\mu_{4r+s}\mu_r\mu_{4s} + 3\mu_{r+4s}\mu_{4r}\mu_s}{n^3}$$

$$+\frac{-9\mu_{4r}\mu_{4s}\mu_r\mu_s + 6\mu_{2r+s}\mu_{r+2s}\mu_{2r}\mu_{2s} + 6\mu_{2r+s}\mu_{2r+2s}\mu_r\mu_{2s} + 6\mu_{r+2s}\mu_{2r+2s}\mu_{2r}\mu_s}{n^3}$$

$$+\frac{6\mu_{3r+s}\mu_{r+s}\mu_r\mu_{3s} + 6\mu_{r+3s}\mu_{r+s}\mu_{3r}\mu_s + 6\mu_{3r+s}\mu_{r+3s}\mu_r\mu_s - 12\mu_{r+2s}\mu_{2r}^2\mu_{2s}\mu_s}{n^3}$$

$$+\frac{-12\mu_{2r+s}\mu_{2r}\mu_r\mu_{2s}^2 - 24\mu_{r+s}\mu_{3r}\mu_{3s}\mu_r\mu_s - 12\mu_{2r+s}\mu_{2r}\mu_r\mu_{2s}^2 - 12\mu_{r+2s}\mu_{2r}^2\mu_{2s}\mu_s}{n^3}$$

$$+\frac{-24\mu_{2r+2s}\mu_{2r}\mu_r\mu_{2s}\mu_s - 24\mu_{3r+s}\mu_r^2\mu_{3s}\mu_s - 24\mu_{r+3s}\mu_{3r}\mu_r\mu_s^2}{n^3}] \tag{96}$$

Lemma 3

a)

$$E(S_n) = 0 \tag{97}$$

b)

$$E(S_n^2) = 1 \tag{98}$$

c)

$$E(S_n^3) = \frac{\sigma^{3(r+s)}}{\sqrt{n}\rho^3(r,s)}(\mu_{3r}\mu_{3s} + 3\mu_{2r+s}\mu_r\mu_{2s} + 3\mu_{2s+r}\mu_{2r}\mu_s - 9\mu_{2r}\mu_r\mu_s\mu_{2s}$$

$$+6\mu_{r+s}^2\mu_r\mu_s - 24\mu_{r+s}\mu_r^2\mu_s^2 + 20\mu_r^3\mu_s^3) \tag{99}$$

d)

$$E(S_nU_n) = \frac{A_1\sigma^{3(r+s)}(\mu_{3r}\mu_{3s} + \mu_{2r}\mu_s\mu_{r+2s} + \mu_r\mu_{2s}\mu_{2r+s} - 3\mu_r\mu_s\mu_{2r}\mu_{2s})}{\rho^3(r,s)}$$

e)

$$E(S_n^2U_n) = O\left(\frac{1}{\sqrt{n}}\right) \tag{100}$$

f)

$$E(S_n^3U_n) = \frac{\sigma^{5(r+s)} * A_1}{\rho^5(r,s)}(27\mu_r^3\mu_s^3\mu_{2r}\mu_{2s} - 9\mu_r^3\mu_s^2\mu_{2r+s}\mu_{2s} - 9\mu_r^2\mu_s^3\mu_{r+2s}\mu_{2r}$$

$$-18\mu_r^2\mu_s^2\mu_{r+s}\mu_{2r}\mu_{2s} + 6\mu_{r+s}\mu_{r+2s}\mu_r\mu_{2r}\mu_s^2 + 6\mu_{r+s}\mu_{2r+s}\mu_r^2\mu_s\mu_{2s}$$

$$+3\mu_{3r}\mu_{3s}\mu_{2r}\mu_{2s} + 6\mu_{r+s}\mu_{3r}\mu_{3s}\mu_r\mu_s + 3\mu_{2r+s}\mu_{2r}\mu_r\mu_{2s}^2 + 3\mu_{r+2s}\mu_{2r}^2\mu_s\mu_{2s}$$

$$-9\mu_{2r}^2\mu_{2s}^2\mu_r\mu_s - 9\mu_r^2\mu_s^2\mu_{3r}\mu_{3s}) + O\left(\frac{1}{n}\right) \tag{101}$$

Lemma 4

a)
$$E^*(V^*(X,r,s)^n) = V(X,r,s)^n \tag{102}$$

b)
$$E^*([V^*(X,r,s)^n]^2) = [V(X,r,s)^n]^2 + \frac{V(X,2r,2s)^n - [V(X,r,s)^n]^2}{n} \tag{103}$$

c)
$$\begin{aligned} E^*([V^*(X,r,s)^n]^3) &= (V(X,r,s)^n)^3 + \frac{3V(X,2r,2s)^n V(X,r,s)^n - 3(V(X,r,s)^n)^3}{n} \\ &+ \frac{V(X,3r,3s)^n - 3V(X,2r,2s)^n V(X,r,s)^n + 2(V(X,r,s)^n)^3}{n^2} \end{aligned} \tag{104}$$

d)
$$\begin{aligned} &E^*([V^*(X,r,s)^n]^4) = (V(X,r,s)^n)^4 \\ &+ \frac{-6(V(X,r,s)^n)^4 + 6V(X,2r,2s)^n(V(X,r,s)^n)^2}{n} \\ &+ \frac{11(V(X,r,s)^n)^4 + 4V(X,3r,3s)^n V(X,r,s)^n - 18V(X,2r,2s)^n(V(X,r,s)^n)^2}{n^2} \\ &+ \frac{3(V(X,2r,2s)^n)^2}{n^2} + \frac{V(X,4r,4s)^n - 6(V(X,r,s)^n)^4}{n^3} \\ &+ \frac{-4V(X,3r,3s)^n V(X,r,s)^n + 12V(X,2r,2s)^n(V(X,r,s)^n)^2 - 3(V(X,2r,2s)^n)^2}{n^3} \end{aligned} \tag{105}$$

e)

$$
\begin{aligned}
&E^*([V^*(X,r,s)^n]^5) = (V(X,r,s)^n)^5 \\
&+\frac{-10(V(X,r,s)^n)^5 + 10V(X,2r,2s)^n(V(X,r,s)^n)^3}{n} \\
&+\frac{35(V(X,r,s)^n)^5 + 10V(X,3r,3s)^n(V(X,r,s)^n)^2}{n^2} \\
&+\frac{15(V(X,2r,2s)^n)^2V(X,r,s)^n - 60V(X,2r,2s)^n(V(X,r,s)^n)^3}{n^2} \\
&+\frac{-50(V(X,r,s)^n)^5 + 5V(X,4r,4s)^nV(X,r,s)^n}{n^3} \\
&+\frac{10V(X,3r,3s)^nV(X,2r,2s)^n - 30V(X,3r,3s)^n(V(X,r,s)^n)^2}{n^3} \\
&+\frac{-45V(X,r,s)^n(V(X,2r,2s)^n)^2 + 110V(X,2r,2s)^n(V(X,r,s)^n)^3}{n^3} \\
&+\frac{V(X,5r,5s)^n + 24(V(X,r,s)^n)^5 - 5V(X,4r,4s)^nV(X,r,s)^n}{n^4} \\
&+\frac{-10V(X,3r,3s)^nV(X,2r,2s)^n + 30(V(X,2r,2s)^n)^2V(X,r,s)^n}{n^4} \\
&+\frac{-60V(X,2r,2s)^n(V(X,r,s)^n)^3 + 20V(X,3r,3s)^n(V(X,r,s)^n)^2}{n^4}
\end{aligned}
\tag{106}
$$

f)

$$
\begin{aligned}
&E^*[V^*(X,r,s)^nV^*(X,2r,2s)^n] = V(X,r,s)^nV(X,2r,2s)^n \\
&+\frac{V(X,3r,3s)^n - V(X,r,s)^nV(X,2r,2s)^n}{n}
\end{aligned}
\tag{107}
$$

g)

$$
\begin{aligned}
&E^*[(V^*(X,r,s)^n)^2V^*(X,2r,2s)^n] = V(X,2r,2s)^n(V(X,r,s)^n)^2 \\
&+\frac{(V(X,2r,2s)^n)^2 + 2V(X,3r,3s)^nV(X,r,s)^n - 3V(X,2r,2s)^n(V(X,r,s)^n)^2}{n} \\
&+\frac{V(X,4r,4s)^n - (V(X,2r,2s)^n)^2 + 2V(X,2r,2s)^n(V(X,r,s)^n)^2 - 2(V(X,2r,2s)^n)^2}{n^2}
\end{aligned}
\tag{108}
$$

h)

$$\begin{aligned}
&E^*[(V^*(X,r,s)^n)^3 V^*(X,2r,2s)^n] = (V(X,r,s)^n)^3 V(X,2r,2s)^n \\
&+\frac{-6(V(X,r,s)^n)^3 V(X,2r,2s)^n + 3(V(X,2r,2s)^n)^2 V(X,r,s)^n}{n} \\
&+\frac{3(V(X,r,s)^n)^2 V(X,3r,3s)^n}{n} + \frac{11(V(X,r,s)^n)^3 V(X,2r,2s)^n}{n^2} \\
&+\frac{3V(X,r,s)^n V(X,4r,4s)^n + 4V(X,2r,2s)^n V(X,3r,3s)^n}{n^2} \\
&+\frac{-9(V(X,2r,2s)^n)^2 V(X,r,s)^n - 9(V(X,r,s)^n)^2 V(X,3r,3s)^n}{n^2} \\
&+\frac{V(X,5r,5s)^n - 6(V(X,r,s)^n)^3 V(X,2r,2s)^n - 3V(X,4r,4s)^n V(X,r,s)^n}{n^3} \\
&+\frac{-4V(X,2r,2s)^n V(X,3r,3s)^n}{n^3} \\
&+\frac{6(V(X,2r,2s)^n)^2 V(X,r,s)^n + 6(V(X,r,s)^n)^2 V(X,3r,3s)^n}{n^3} \qquad (109)
\end{aligned}$$

i)

$$E^*(\rho^*(r,s)^n) \xrightarrow{\mathcal{P}} \rho^*(r,s) \qquad (110)$$

j)

$$Var^*(\rho^*(r,s)^n) \xrightarrow{\mathcal{P}} 0 \qquad (111)$$

Lemma 5

a)

$$E^*(S_n^*) = 0$$

b)

$$E^*(S_n^{*2}) = 1$$

c)

$$E^*(S_n^{*3}) = \frac{V(X,3r,3s)^n - 3V(X,2r,2s)^n V(X,r,s)^n + 2(V(X,r,s)^n)^3}{\sqrt{n}\rho^{*3}(r,s)}$$

d)

$$E^*(S_n^* U_n^*) = \frac{V(X,3r,3s)^n - 3V(X,r,s)^n V(X,2r,2s)^n + 2(V(X,r,s)^n)^3}{\rho^{*3}(r,s)} + O_P\left(\frac{1}{n}\right)$$

e)

$$E^*(S_n^{*2}U_n^*) = O_P\left(\frac{1}{\sqrt{n}}\right)$$

f)

$$E^*(S_n^{*3}U_n^*) = 3\frac{V(X,3r,3s)^n - 3V(X,r,s)^n V(X,2r,2s)^n + 2(V(X,r,s)^n)^3}{\rho^{*3}(r,s)} + O_P\left(\frac{1}{n}\right)$$

g)

$$I_1^1 = \frac{V(X,3r,3s)^n - V(X,r,s)^n V(X,2r,2s)^n}{n}$$

h)

$$I_1^2 = \frac{V(X,4r,4s)^n - (V(X,2r,2s)^n)^2 - 2V(X,3r,3s)^n V(X,r,s)^n}{n^2} + \frac{2V(X,2r,2s)^n (V(X,r,s)^n)^2}{n^2} \qquad (112)$$

i)

$$I_1^3 = 3\frac{(V(X,3r,3s)^n - V(X,2r,2s)^n V(X,r,s)^n)(V(X,2r,2s)^n - [V(X,r,s)^n]^2)}{n^2} + O_P\left(\frac{1}{n^3}\right) \qquad (113)$$

j)

$$I_2^1 = \frac{V(X,3r,3s)^n - 3V(X,r,s)^n V(X,2r,2s)^n + 2(V(X,r,s)^n)^3}{n^2}$$

k)

$$I_2^2 = \frac{3[V(X,2r,2s)^n - (V(X,r,s)^n)^2]^2}{n^2} + \frac{V(X,4r,4s)^n - 4V(X,3r,3s)^n V(X,r,s)^n + 12V(X,2r,2s)^n (V(X,r,s)^n)^2}{n^3} + \frac{-6(V(X,r,s)^n)^4 - 3(V(X,2r,2s)^n)^2}{n^3} \qquad (114)$$

l)

$$I_2^3 = \frac{10[V(X,3r,3s)^n - 3V(X,2r,2s)^n V(X,r,s)^n + 2(V(X,r,s)^n)^3]}{n^3} * \frac{[V(X,2r,2s)^n - (V(X,r,s)^n)^2]}{n^3} + O_P\left(\frac{1}{n^4}\right) \qquad (115)$$

m)

$$I_3^1 = \frac{-2V(X,r,s)^n}{n}\left(V(X,2r,2s)^n - (V(X,r,s)^n)^2\right)$$

n)

$$I_3^2 = \frac{-2V(X,r,s)^n}{n^2}[V(X,3r,3s)^n - 3V(X,2r,2s)^n V(X,r,s)^n + 2(V(X,r,s)^n)^3]$$

o)

$$I_3^3 = \frac{-6V(X,r,s)^n}{n^2}[V(X,2r,2s)^n - (V(X,r,s)^n)^2]^2 + O_P\left(\frac{1}{n^3}\right)$$

Remark 7
The last 9 I-terms come from the following relation

$$\begin{aligned}
&\rho^{*2}(r,s)^n - \rho^{*2}(r,s) = V^*(X,2r,2s)^n - (V^*(X,r,s)^n)^2 - V(X,2r,2s)^n \\
&+(V(X,r,s)^n)^2 = V^*(X,2r,2s)^n - V(X,2r,2s)^n - [V^*(X,r,s)^n - V(X,r,s)^n]^2 \\
&-2V(X,r,s)^n[V^*(X,r,s)^n - V(X,r,s)^n]. \qquad (116)
\end{aligned}$$

As immediate consequence we obtain

$$\begin{aligned}
&E^*[(V^*(X,r,s)^n - V(X,r,s)^n)^q(\rho^{*2}(r,s)^n - \rho^{*2}(r,s))] \\
&= E^*[(V^*(X,r,s)^n - V(X,r,s)^n)^q(V^*(X,2r,2s)^n - V(X,2r,2s)^n)] \\
&-E^*[(V^*(X,r,s)^n - V(X,r,s)^n)^{2+q}] \\
&-2V(X,r,s)^n E^*[(V^*(X,r,s)^n - V(X,r,s)^n)^{1+q}] \\
&= I_1^q - I_2^q - I_3^q. \qquad (117)
\end{aligned}$$

Lemma 6

a)

$$E^{WB}(S_n^{WB}) = 0$$

b)

$$E^{WB}(S_n^{2WB}) = 1$$

c)

$$E^{WB}(S_n^{3WB}) = \frac{1}{\sqrt{n}}\frac{V(X,3r,3s)^n}{(V(X,2r,2s)^n)^{\frac{3}{2}}}\frac{\mu_6^{WB} - 3\mu_4^{WB}\mu_2^{WB} + 2(\mu_2^{WB})^3}{(\mu_4^{WB} - (\mu_2^{WB})^2)^{\frac{3}{2}}}$$

d)

$$\begin{aligned}
&E^{WB}(S_n^{WB}U_n^{WB}) = n^{\frac{3r+3s}{2}-1}(\mu_4^{WB} - (\mu_2^{WB})^2) * \\
&\frac{\frac{\mu_6^{WB}}{\mu_4^{WB}}\sum_{i=1}^n \mid \Delta_i^n X \mid^{3r} \mid \Delta_{i+1}^n X \mid^{3s} - \mu_2^{WB}\sum_{i=1}^n \mid \Delta_i^n X \mid^{3r} \mid \Delta_{i+1}^n X \mid^{3s}}{\rho^{WB3}(r,s)}
\end{aligned} \qquad (118)$$

e)
$$E^{WB}\left(\frac{S_n^{2WB}U_n^{WB}}{\sqrt{n}}\right)=O_P\left(\frac{1}{n}\right)$$

f)
$$E^{WB}\left(\frac{S_n^{3WB}U_n^{WB}}{\sqrt{n}}\right)=\frac{1}{\sqrt{n}}\frac{3(\mu_6^{WB}-\mu_4^{WB}\mu_2^{WB})}{\mu_4^{WB}(\mu_4^{WB}-(\mu_2^{WB})^2)^{\frac{1}{2}}}\frac{V(X,3r,3s)^n}{(V(X,2r,2s)^n)^{\frac{3}{2}}}$$

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Number of Data	Nor	Nor-log	i.i.d. Boot	i.i.d. Boot-log	WB	WB-log
50	73,8	81,1	86	93,9	80,2	87,6
100	80,1	85,2	89,3	94,2	85,1	91,3
200	85,9	88,8	91,8	95	90,1	93,7
500	89,6	92	93,7	95,3	92,9	94,8
1000	88,7	90,4	92,5	93,8	91,7	93,4

Table 1: *Results for simulated 1-sided 95%-confidence sets and r=4, s=0 with constant volatility and* $\mu = 0$.

Number of Data	Nor	Nor-log	i.i.d. Boot	i.i.d. Boot-log	WB	WB-log
50	78,8	83,8	90,1	92,5	80,8	85,4
100	84,2	87,8	91,6	93	87,5	89,2
200	88,9	91,4	93,6	93,2	91,8	92,1
500	92,2	93,3	93,8	94,5	93,3	93,9
1000	91,7	92,8	93,1	94,2	92,8	94,3

Table 2: *Results for simulated 2-sided 95%-confidence sets and r=4, s=0 with constant volatility and* $\mu = 0$.

Number of Data	Nor	Nor-log	i.i.d. Boot	i.i.d. Boot-log	WB	WB-log
50	74,8	82,4	87,7	95,5	81,2	90,5
100	81,4	86,6	90,1	94,4	86,6	91,9
200	84,9	88,2	90,4	94	89,4	93,1
500	89	91,1	93,1	95,2	92	94,4
1000	90,1	91,1	92,6	94,7	92,3	94

Table 3: *Results for simulated 1-sided 95%-confidence sets and r=4, s=0 with the 2-factor-model and* $\mu = 0$.

Number of Data	Nor	Nor-log	i.i.d. Boot	i.i.d. Boot-log	WB	WB-log
50	79,5	85,3	92,2	92,6	82,4	86
100	85,2	88,9	93,2	93,7	88,6	90,3
200	88,1	89,1	92,3	92,8	90	90,2
500	91,4	93,1	94,5	94,9	93,4	94,6
1000	92,1	93,5	94,2	94,3	93,4	93,7

Table 4: *Results for simulated 2-sided 95%-confidence sets and r=4, s=0 with the 2-factor-model and* $\mu = 0$.

Number of Data	Nor	Nor-log	i.i.d. Boot	i.i.d. Boot-log	WB	WB-log
50	76,1	82,7	88	94,3	81,4	89,9
100	82,6	87,5	91,3	95,3	88,4	94
200	89,2	88,4	91,3	95,2	89,7	92,6
500	88,9	91,4	93,2	95,9	92,8	95,2
1000	92,3	93,6	95	95,6	94,4	95,8

Table 5: *Results for simulated 1-sided 95%-confidence sets and r=4, s=0 with constant volatility and* $\mu = 0,03$.

Number of Data	Nor	Nor-log	i.i.d. Boot	i.i.d. Boot-log	WB	WB-log
50	79,9	86,3	91,5	94	82,6	87,5
100	86	90,7	93,6	93,4	89,7	91
200	88,8	90,9	93,3	94,8	91	93,3
500	92	93,8	95,2	93,4	94	93
1000	94,4	94,9	94,3	94	94,5	93,9

Table 6: *Results for simulated 2-sided 95%-confidence sets and r=4, s=0 with constant volatility and* $\mu = 0,03$.

Number of Data	Nor	Nor-log	i.i.d. Boot	i.i.d. Boot-log	WB	WB-log
50	75	82,4	88,6	94,7	81	89,1
100	82,6	87,8	92	95,2	88,1	93,2
200	84,8	88,8	91,7	95	89,9	93,5
500	87,9	90,3	92,8	95,1	91,4	94,4
1000	91,3	92,9	94,4	95,7	94,3	95,2

Table 7: *Results for simulated 1-sided 95%-confidence sets and r=4, s=0 with the 2-factor-model and* $\mu = 0,03$.

Number of Data	Nor	Nor-log	i.i.d. Boot	i.i.d. Boot-log	WB	WB-log
50	79,3	85,8	92,6	93,7	81,8	87
100	86,4	90,4	94	92,3	89,4	90,3
200	88,8	91,3	94,2	94	91,7	92,6
500	91	92,4	94,5	94,2	93,1	93,5
1000	93	93,2	93,8	94,1	92,9	93,5

Table 8: *Results for simulated 2-sided 95%-confidence sets and r=4, s=0 with the 2-factor-model and* $\mu = 0,03$.

Number of Data	Nor	Nor-log	i.i.d. Boot	i.i.d. Boot-log	WB	WB-log
50	88,1	92,6	90,8	93,7	90,7	94,2
100	88,1	90,9	90,2	92,9	91,2	93,4
200	91,6	93,6	92,1	93,7	92,8	94,5
500	91,6	92,8	91,9	93,4	92,6	94,1
1000	92,7	93,6	93,3	94	93,8	94,1

Table 9: *Results for simulated 1-sided 95%-confidence sets and r=1, s=1 with constant volatility and* $\mu = 0$.

Number of Data	Nor	Nor-log	i.i.d. Boot	i.i.d. Boot-log	WB	WB-log
50	91,8	94,1	93,4	95,2	93,1	95
100	91,1	93,6	92,6	94,2	92,8	93,8
200	93,9	94,9	94	94,9	96,6	95,1
500	93,8	94,5	93,4	94,3	93,6	94,4
1000	94,6	94,8	94,2	94,2	94,4	94,4

Table 10: *Results for simulated 2-sided 95%-confidence sets and r=1, s=1 with constant volatility and* $\mu = 0$.

Number of Data	Nor	Nor-log	i.i.d. Boot	i.i.d. Boot-log	WB	WB-log
50	87,8	91,8	89,7	93,3	90,4	94,1
100	89,2	91,8	89,9	92,4	90,8	94,2
200	90	92,1	91	94,3	91,8	94,6
500	92,4	93,3	93	94	93,3	94,3
1000	93,3	94,2	94	94,4	94,2	94,8

Table 11: *Results for simulated 1-sided 95%-confidence sets and r=1, s=1 with the 2-factor-model and* $\mu = 0$.

Number of Data	Nor	Nor-log	i.i.d. Boot	i.i.d. Boot-log	WB	WB-log
50	90,7	93,5	93,3	94,1	91,9	93,3
100	92,4	93,6	93,7	94,6	93,4	94,7
200	93,7	94,7	94,3	95,5	95,2	95
500	93,8	95	94,7	95,4	94,8	95,3
1000	93,7	93,7	93,5	93,7	93,4	93,4

Table 12: *Results for simulated 2-sided 95%-confidence sets and r=1, s=1 with the 2-factor-model and* $\mu = 0$.

Number of Data	Nor	Nor-log	i.i.d. Boot	i.i.d. Boot-log	WB	WB-log
50	88,3	92,6	90,3	93,3	90	93,4
100	89,5	92,8	91,8	94,3	93	95,2
200	90,4	92,5	92,1	93,5	92,1	93,9
500	92,4	93,6	93,5	94,2	93,4	94,8
1000	92,9	93,5	92,5	93,6	93,1	93,9

Table 13: *Results for simulated 1-sided 95%-confidence sets and r=1, s=1 with constant volatility and* $\mu = 0,03$.

Number of Data	Nor	Nor-log	i.i.d. Boot	i.i.d. Boot-log	WB	WB-log
50	92,2	94,2	93,8	95,2	92,2	94,6
100	93,5	94,6	94,2	95,2	94,8	95,4
200	93,7	94,3	94,7	95	94,7	95,2
500	95,1	95,7	95,8	95,8	95,5	95,7
1000	93,2	93,3	93,9	93,9	93,6	93,8

Table 14: *Results for simulated 2-sided 95%-confidence sets and r=1, s=1 with constant volatility and* $\mu = 0,03$.

Number of Data	Nor	Nor-log	i.i.d. Boot	i.i.d. Boot-log	WB	WB-log
50	86,2	91,6	89,7	93,8	89,2	93,1
100	90,1	92,1	90,7	93,9	91,3	93,7
200	91,9	94,1	93,4	95,3	93,8	95,5
500	92,2	93,2	92,9	94,3	93,2	94,7
1000	92,9	93,5	93,1	94	93,4	94,1

Table 15: *Results for simulated 1-sided 95%-confidence sets and r=1, s=1 with the 2-factor-model and* $\mu = 0,03$.

Number of Data	Nor	Nor-log	i.i.d. Boot	i.i.d. Boot-log	WB	WB-log
50	91	94,7	94,1	96,3	93	94,5
100	92,5	95	94,7	95,6	93,9	95,5
200	94,5	95,4	94,4	95	95,2	94,7
500	94,8	95,4	95,4	95,4	95,4	95,5
1000	94,7	95	95,1	95,2	94,5	94,9

Table 16: *Results for simulated 2-sided 95%-confidence sets and r=1, s=1 with the 2-factor-model and $\mu = 0,03$.*

r/s	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	0,16	0	0,09	0,15	0,19	0,21	0,22	0,22	0,22	0,22
1	0,291	0,2	0,05	0,14	0,32	0,47	0,6	0,69	0,75	0,79
2	0,2	0,2	0,05	0,21	0,54	0,88	1,2	1,49	1,73	1,92
3	0,05	0,05	0,13	0,49	0,99	1,57	2,2	2,8	3,38	3,91
4	0,14	0,21	0,49	1,01	1,76	2,69	3,75	4,87	6,02	7,14
5	0,32	0,54	0,99	1,76	2,87	4,3	6	7,91	9,95	12,08
6	0,47	0,88	1,57	2,69	4,3	6,42	9,03	12,08	15,51	19,24
7	0,6	1,2	2,2	3,75	6	9,03	12,88	17,54	22,97	29,1
8	0,69	1,49	2,81	4,87	7,91	12,08	17,54	24,36	32,56	42,13
9	0,75	1,73	3,38	6,02	9,95	15,51	22,97	32,56	44,45	58,75
10	0,79	1,92	3,91	7,13	12,08	19,24	29,1	42,13	58,75	79,3
r/s	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	0,7	1,4	2,7	4,9	9	16,4	30	55	100,7	184,9
1	1,6	2,7	4,5	7,9	14,2	25,8	47,4	87,1	160,1	294,4
2	2,7	4,6	8	14,1	25,4	46,4	85,3	157	289	531,7
3	4,5	8	14,1	25,4	46,1	84,6	155,6	286,5	527,5	970,8
4	7,9	14,1	25,4	46,1	84,2	154,7	284,9	524,8	966,3	1778
5	14,1	25,4	46,1	84,2	154,5	284,1	523,1	963,2	1773	3262
6	25,8	46,4	84,6	154,7	284,1	522,5	961,6	1770	3256	5988
7	47,4	85,3	155,6	284,9	523,1	961,6	1768	3253	5982	10998
8	87,11	157	284,5	324,8	963,2	1770	3253	5979	10992	20203
9	160,1	289	527,5	966,3	1773	3256	5982	10992	20198	37115
10	294,4	531,7	970,8	1778	3262	5988	10998	20203	37115	68187

Table 17: *The upper panel of this table shows some values of $|k_1 - k_1^*|$ as given in Lemma 1 for different combinations of r and s. The lower panel of this table shows some values of k_1 as given in Lemma 1 for different combinations of r and s.*

r/s	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	0,99	0	0,56	0,92	1,15	1,28	1,34	1,36	1,34	1,29
1	0,85	0,2	0,46	0,96	1,27	1,43	1,48	1,47	1,43	1,36
2	0,2	0,03	0,4	0,88	1,24	1,45	1,52	1,5	1,44	1,36
3	0,46	0,4	0,59	0,93	1,25	1,45	1,53	1,51	1,44	1,34
4	0,96	0,88	0,93	1,12	1,34	1,5	1,56	1,54	1,45	1,34
5	1,27	1,24	1,25	1,34	1,48	1,58	1,61	1,57	1,48	1,35
6	1,43	1,45	1,45	1,5	1,58	1,65	1,66	1,6	1,5	1,38
7	1,48	1,52	1,53	1,56	1,61	1,66	1,66	1,61	1,52	1,39
8	1,47	1,5	1,51	1,54	1,57	1,61	1,61	1,58	1,5	1,39
9	1,43	1,44	1,44	1,45	1,48	1,51	1,52	1,5	1,45	1,36
10	1,36	1,36	1,34	1,34	1,35	1,38	1,39	1,39	1,36	1,3
r/s	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	-3	-6	-10	-20	-35	-65	-119	-219	-402	-739
1	-6	-10	-17	-31	-57	-104	-191	-350	-642	-1179
2	-10	-18	-31	-56	-103	-188	-345	-633	-1162	-2133
3	-17	-31	-56	103	-187	-343	-630	-1156	-2122	-3897
4	-31	-56	-103	-187	-343	-628	-1153	-2117	-3888	-7140
5	-57	-103	-187	-343	-628	-1152	-2115	-3883	-7130	-13094
6	-104	-188	-343	-628	-1152	-2114	-3881	-7125	-13084	-24028
7	-191	-345	-630	-1153	-2115	-3881	-7124	-13079	-24017	-44106
8	-350	-633	-1156	-2117	-3883	-7125	-13079	-24014	-44095	-80977
9	-642	-1162	-2122	-3888	-7130	-13084	-24017	-44095	-80970	-148695
10	-1179	-2133	-3897	-7140	-13094	-24028	-44106	-80978	-148695	-273064

Table 18: *The upper panel of this table shows some values of* $|k_3 - k_3^*|$ *as given in Lemma 1 for different combinations of r and s. The lower panel of this table shows some values of* k_3 *as given in Lemma 1 for different combinations of r and s.*