A Note on the Central Limit Theorem for Bipower Variation of General Functions *

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Abstract

In this paper we present a central limit theorem for general functions of the increments of Brownian semimartingales. This provides a natural extension of the results derived in Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006), who showed the central limit theorem for even functions. We prove an infeasible central limit theorem for general functions and state some assumptions under which a feasible version of our results can be obtained. Finally, we present some examples from the literature to which our theory can be applied.

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1 Introduction

We consider a d-dimensional semimartingale, defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, of the form

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s \ , \tag{1.1}$$

where W denotes a d'-dimensional Brownian motion, a is a d-dimensional locally bounded and predictable drift process, and σ is a $\mathbb{R}^{d\times d'}$-valued càdlàg volatility process. Models of the type (1.1) and their extensions are widely used in mathematical finance to capture the dynamics of stock prices or interest rates.

Recently, the concept of *realised bipower variation* has built a non-parametric framework for backing out several variational measures of volatility, which has led to a new development in econometrics. Realised bipower variation, which is given by

$$V(X,r,l)_t^n = n^{\frac{r+l}{2}-1} \sum_{i=1}^{n-1} |\Delta_i^n X|^r |\Delta_{i+1}^n X|^l \ , \tag{1.2}$$

with $\Delta_i^n X = X_{i/n} - X_{(i-1)/n}$ and $r, l \geq 0$, provides a whole variety of estimators for different (integrated) powers of volatility (in (1.2) the process X is assumed to be one-dimensional). An important special case of the class (1.2) is the *realised volatilty*

$$\sum_{i=1}^{[nt]} |\Delta_i^n X|^2 \ ,$$

which is a consistent estimator of the quadratic variation of X, i.e.

$$IV_t = \int_0^t \sigma_s^2 ds \ ,$$

which is often referred to as integrated volatility in the econometric literature.

Statistics of the form (1.2) have been intensively studied in the last years. Theoretical and empirical properties of the realised volatility have been discussed in numerous articles (see Jacod (1994), Jacod & Protter (1998), Andersen, Bollerslev, Diebold & Labys (1998), Andersen, Bollerslev & Diebold (2006), Barndorff-Nielsen & Shephard (2002, 2004a) among many others). Let us also mention the work of Christensen & Podolskij (2006a,b) who have derived the asymptotic distribution theory for the quantities of the type (1.2) when returns of X are replaced by ranges of X. Asymptotic properties of realised bipower variation have been used to construct tests for jumps (see Barndorff-Nielsen & Shephard (2006), Christensen & Podolskij (2006b) or Ait-Sahalia & Jacod (2006)) or to provide goodness-of-fit tests for the parametric form of the volatility function in stochastic differential equations (see Dette, Podolskij & Vetter (2006) or Dette & Podolskij (2006)).

The central object of our study are the processes of the form

$$V(X,g,h)_t^n = \frac{1}{n} \sum_{i=1}^{[nt]} g(\sqrt{n}\Delta_i^n X) h(\sqrt{n}\Delta_{i+1}^n X) \ , \tag{1.3}$$

where g, h are two maps on $I\!R^d$, taking values in $I\!R^{d_1 \times d_2}$ and $I\!R^{d_2 \times d_3}$, respectively. Obviously, processes of the type (1.3) are a generalisation of (1.2) (they are sometimes called *generalised bipower variation*). Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006) showed the consistency of $V(X, g, h)_t^n$ and derived a (stable) central limit theorem for its standardised version. However, the central limit theorem depends crucially on the assumption that both functions g and h are even (i.e. $g(x) = g(-x)$, $h(x) = h(-x)$ for all $x \in I\!R^d$).

In this paper we prove the central limit theorem for the class (1.3) for general functions g and h. It turns out that the properly normalised version of $V(X, g, h)_t^n$ converges stably in law to a mixed normal process with drift. More precisely, the limiting process is (non-centered) Gaussian conditionally on the σ-algebra $\mathcal{F}$. Although this central limit theorem is a nice probabilistic result, it is in general infeasible. In the next step we provide some conditions on g and h under which we obtain a feasible central limit theorem. In order to illustrate our theoretical results we state the asymptotic theory for the realised bipower variation, *realised covariation* of X (the multivariate version of realised volatility), *realised autocovariance* of X (this statistic appears in Zhang, Mykland & Ait-Sahalia (2005) and Barndorff-Nielsen, Hansen, Lunde & Shephard (2006)), the normalised sum of the third power of returns (this quantity plays a crucial role in Jiang & Oomen (2006)) and for some other statistics.

This article is structured as follows. In Chapter 2 we present the main theoretical results. We discuss the above-mentioned examples in Chapter 3. Finally, we state the proofs in the Appendix.

2 Central limit theorem

Before we state the main results, we introduce some notations. Below $U = (U^1, \ldots, U^{d'})^T$ is a d'-dimensional standard normal, f is a real-valued function on $I\!R^d$, Σ is a $d \times d'$-dimensional matrix and $W = (W^1, \ldots, W^{d'})^T$ is a d'-dimesional Brownian motion. For $1 \leq k, s \leq d'$ we define

$$\rho_\Sigma(f) = E[f(\Sigma U)] \ ,$$

$$\rho_\Sigma^{(k)}(f) = E[f(\Sigma U) U^k] \ ,$$

$$\tilde{\rho}_\Sigma^{(sk)}(f) = E[f(\Sigma W_1) \int_0^1 W_u^s \, dW_u^k]. \tag{2.1}$$

Note that due to the symmetry of the standard normal distribution we have $\rho_\Sigma(f) = \tilde{\rho}_\Sigma^{(sk)}(f) = 0$ for all odd functions f, whereas the identity $\rho_\Sigma^{(k)}(f) = 0$ holds for all even functions f. Further, we set

$$\mu_r = E[|z|^r], \qquad z \sim N(0, 1). \tag{2.2}$$

The following Theorem, which has been derived in Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006), gives the probability limit of the sequence $V(X, g, h)_t^n$ defined by (1.3).

Theorem 1 *Assume that the process X is of the form (1.1) and the functions g, h are continuous with at most polynomial growth. Then we have*

$$V(X,g,h)_t^n \xrightarrow{P} V(X,g,h)_t = \int_0^t \rho_{\sigma_u}(g)\rho_{\sigma_u}(h)du \ , \tag{2.3}$$

where the convergence holds locally uniform in t.

When all processes are one-dimensional and $g(x) = |x|^r$, $h(x) = |x|^l$ $(r, l \geq 0)$ it follows that

$$\rho_{\sigma_u}(g) = \mu_r |\sigma_u|^r \ , \qquad \rho_{\sigma_u}(g) = \mu_l |\sigma_u|^l \ ,$$

and consequently we obtain a well-known result (see, for instance, Barndorff-Nielsen & Shephard (2004a)) for the realised bipower variation

$$V(X,r,l)_t^n \xrightarrow{P} V(X,r,l)_t = \mu_r \mu_l \int_0^t |\sigma_u|^{r+l} du \ , \tag{2.4}$$

where the convergence holds locally uniform in t. Recall that the limit process $V(X,g,h)_t$ equals 0 for all t if g or h is an odd function.

Next, we demonstrate a (stable) central limit theorem for the sequence of processes $\sqrt{n}(V(X,g,h)_t^n - V(X,g,h)_t)$. For this purpose we require a stronger condition on the volatility process σ:

(V): The volatility function σ satisfies the equation

$$\sigma_t = \sigma_0 + \int_0^t a'_s ds + \int_0^t \sigma'_s dW_s + \int_0^t v'_s dV_s. \tag{2.5}$$

Here a',σ' and v' are adapted càdlàg processes, with a' also being predictable and locally bounded, and V is a new Brownian motion independent of W.

Assumption (V) is a standard sufficient condition that is required for the proof of the central limit theorem (see, e.g., Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006), Christensen & Podolskij (2006a,b) or Podolskij & Vetter (2006)). When the process X is a unique strong solution of a stochastic differential equation and $\sigma_s = \sigma(s, X_s) \in C^{1,2}([0,t])$ (i.e. $\sigma(s,x)$ is once continuously differentiable in s and twice continuously differentiable in x) then the representation (2.5) (with $v'(s) = 0$ for all s) is a simple consequence of the Ito-formula. For this reason assumption (V) is satisfied for many widely used continuous time models (see Black & Scholes (1973), Vasicek (1977), Cox, Ingersoll & Ross (1980) or Chan, Karolyi, Longstaff & Sanders (1992) among others). Note also that the assumption (V) on the volatility process σ can be weakened in order to allow σ to have jumps (see Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006) or Ait-Sahalia & Jacod (2006) for more details). In that case further truncation techniques for the jump part of σ are required to prove the desired central limit theorem.

In the main result of this paper we use the concept of stable convergence. Let us briefly recall the definition. A sequence of random processes Y^n converges stably in law with limit Y (throughout this paper we write $Y^n \xrightarrow{\mathcal{D}_{st}} Y$), defined on an appropriate extension $(\Omega', \mathcal{F}', P')$ of the probability space $(\Omega, \mathcal{F}, P)$, if and only if for any $\mathcal{F}$-measurable and bounded random variable Z and any bounded and continuous function f on the space of all càdlàg functions (endowed with the Skorokhod topology) the convergence

$$\lim_{n\to\infty} E[Zf(Y^n)] \;=\; E[Zf(Y)]$$

holds. This is obviously a slightly stronger mode of convergence than weak convergence (see Renyi (1963), Aldous & Eagleson (1978) or Jacod & Shiryaev (2003) for more details on stable convergence).

Now we present a stable central limit theorem for the class $\sqrt{n}(V(X,g,h)_t^n - V(X,g,h)_t)$.

Theorem 2 *Assume that X is of the form (1.1) and condition (V) is satisfied. If further g and h are continuously differentiable with g, h, $\frac{\partial g^{lk}}{\partial r}$ and $\frac{\partial h^{lk}}{\partial r}$ being of at most polynomial growth (for all l,k,r), we obtain the stable convergence*

$$\sqrt{n}(V(X,g,h)_t^n - V(X,g,h)_t) \xrightarrow{\mathcal{D}_{st}} U(g,h)_t = \int_0^t \alpha_u(1)du + \int_0^t \alpha_u(2)dW_u + \int_0^t \alpha_u(3)dW_u' \;, \tag{2.6}$$

where W' is a d_1d_3-dimensional Brownian motion which is defined on an extension of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and is independent of the σ-field $\mathcal{F}$, $\alpha(1)$, $\alpha(2)$ and $\alpha(3)$ are d_1d_3-, $d_1d_3\times d'$- and $d_1d_3\times d_1d_3$-dimensional processes, respectively, defined componentwise by

$$\begin{aligned}
\alpha_u^{jk}(1) \;&=\; \sum_{l=1}^{d_2}\sum_{r=1}^{d}\sum_{s=1}^{d'}\sum_{i=1}^{d'} \Big\{ \sigma_u'^{rsi}\rho_{\sigma_u}^{(s)}(g^{jl})\rho_{\sigma_u}^{(s)}(\frac{\partial h^{lk}}{\partial r}) + \sigma_u'^{rsi}\rho_{\sigma_u}(g^{jl})\tilde{\rho}_{\sigma_u}^{(si)}(\frac{\partial h^{lk}}{\partial r}) \\
&+\; a_u^r\rho_{\sigma_u}(\frac{\partial g^{jl}}{\partial r})\rho_{\sigma_u}(h^{lk}) + \sigma_u'^{rsi}\rho_{\sigma_u}(h^{lk})\tilde{\rho}_{\sigma_u}^{(si)}(\frac{\partial g^{jl}}{\partial r})\Big\} \\
&+\; \sum_{l=1}^{d_2}\sum_{r=1}^{d} a_{\sigma_u}^r\rho_{\sigma_u}(g^{jl})\rho_{\sigma_u}(\frac{\partial h^{lk}}{\partial r}) \;,
\end{aligned} \tag{2.7}$$

$$\alpha_u^{jkr}(2) \;=\; \sum_{l=1}^{d_2} \rho_{\sigma_u}(g^{jl})\rho_{\sigma_u}^{(r)}(h^{lk}) + \rho_{\sigma_u}(h^{lk})\rho_{\sigma_u}^{(r)}(g^{jl}) \;, \tag{2.8}$$

$$\alpha_u(3) \;=\; \Big(A_u - \alpha_u(2)\alpha_u(2)^T\Big)^{\frac{1}{2}} \;, \tag{2.9}$$

and the $d_1d_3\times d_1d_3$-dimensional process A is given by

$$\begin{aligned}
A_u^{jk,j'k'} \;&=\; \sum_{l,l'=1}^{d_2} \Big\{\rho_{\sigma_u}(g^{jl}g^{j'l'})\rho_{\sigma_u}(h^{lk}h^{l'k'}) + \rho_{\sigma_u}(g^{jl})\rho_{\sigma_u}(h^{l'k'})\rho_{\sigma_u}(g^{j'l'}h^{lk}) \\
&+\; \rho_{\sigma_u}(g^{j'l'})\rho_{\sigma_u}(h^{lk})\rho_{\sigma_u}(g^{jl}h^{l'k'}) - 3\rho_{\sigma_u}(g^{jl})\rho_{\sigma_u}(g^{j'l'})\rho_{\sigma_u}(h^{lk})\rho_{\sigma_u}(h^{l'k'})\Big\}.
\end{aligned} \tag{2.10}$$

Jacod (1994) was the first who proved the stable convergence of the type (2.6) for $d_2 = d_3$ and $h = id_{\mathbb{R}^{d_2}}$. Let us also mention the stable central limit theorem for sum of semimartingale differences derived in Jacod (1997) (see also Jacod & Shiryaev (2003)) which is absolutely crucial for many problems in the high-frequency framework (in fact, we use this result to prove Theorem 2).

Remark 1 *Note that the differentiability assumption on g and h can be slightly weakened (see Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006) for more details). This is important for the derivation of the stable central limit theorem for the realised bipower variation with $g(x) = |x|^r$, $h(x) = |x|^l$ and $r \in (0,1)$ or $l \in (0,1)$ (these functions are obviously not differentiable at 0). However, we restrict ourselves to the case where g and h are both continuously differentiable for the sake of simplicity.*

Note that conditionally on $\mathcal{F}$ the limit process $U(g,h)_t$ is (non-centered) Gaussian (since the first two summands in (2.6) are measurable with respect to $\mathcal{F}$). Furthermore, the quadratic covariation of $U(g,h)_t$ equals $\int_0^t A_u du$. Even though Theorem 2 is an interesting probabilistic result, it is in general infeasible due to the appearance of the process σ' and the drift a in the limit. However, when all components of g and h are even the limit process $U(g,h)_t$ has a simpler form. In that case the functions $\frac{\partial g^{kl}}{\partial r}$ and $\frac{\partial h^{kl}}{\partial r}$ are odd for all k, l, r. Recall that $\rho_\Sigma(f) = \tilde{\rho}_\Sigma^{(sk)}(f) = 0$ for all odd functions f and $\rho_\Sigma^{(k)}(f) = 0$ for all even functions f, and consequently we obtain $\alpha_u(1) = \alpha_u(2) = 0$ for all u. In fact, $U(g,h)_t$ becomes a mixed normal process and a standard central limit theorem (for fixed t) can be obtained (see the examples in Section 3 or Barndorff-Nielsen, Graversen, Jacod & Shephard (2006) for more details).

For the sake of simplicity let us also demonstrate Theorem 2 for the one-dimensional case.

Corollary 1 *Assume that X is of the form (1.1), condition (V) is satisfied and all processes are one-dimensional. If further g and h are continuously differentiable with g, h, ∇g and ∇h being of at most polynomial growth, we obtain the stable convergence*

$$\sqrt{n}(V(X,g,h)_t^n - V(X,g,h)_t) \xrightarrow{\mathcal{D}_{st}} U(g,h)_t = \int_0^t \alpha_u(1)du + \int_0^t \alpha_u(2)dW_u + \int_0^t \alpha_u(3)dW'_u , \tag{2.11}$$

where W' is a 1-dimensional Brownian motion which is defined on an extension of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and is independent of the σ-field $\mathcal{F}$, $\alpha(1)$, $\alpha(2)$ and $\alpha(3)$ are *defined by*

$$\begin{aligned}\alpha_u(1) &= \sigma'_u\Big\{\rho^{(1)}_{\sigma_u}(g)\rho^{(1)}_{\sigma_u}(\nabla h) + \rho_{\sigma_u}(g)\tilde{\rho}^{(11)}_{\sigma_u}(\nabla h) + \rho_{\sigma_u}(h)\tilde{\rho}^{(11)}_{\sigma_u}(\nabla g)\Big\} \\ &+ a_u\Big\{\rho_{\sigma_u}(\nabla g)\rho_{\sigma_u}(h) + \rho_{\sigma_u}(\nabla h)\rho_{\sigma_u}(g)\Big\}\ , \\ \alpha_u(2) &= \rho_{\sigma_u}(g)\rho^{(1)}_{\sigma_u}(h) + \rho_{\sigma_u}(h)\rho^{(1)}_{\sigma_u}(g)\ , \qquad (2.12) \\ \alpha_u(3) &= \Big(A_u - \alpha^2_u(2)\Big)^{\frac{1}{2}}\ , \qquad (2.13)\end{aligned}$$

and the process A is given by

$$A_u = \rho_{\sigma_u}(g^2)\rho_{\sigma_u}(h^2) + 2\rho_{\sigma_u}(g)\rho_{\sigma_u}(h)\rho_{\sigma_u}(gh) - 3\rho^2_{\sigma_u}(g)\rho^2_{\sigma_u}(h). \qquad (2.14)$$

As already mentioned the limit process in (2.11) is mixed normal if the functions g and h are both even. Interestingly, this is also true when both g and h are odd. This fact can be easily deduced by observing that in this case we have $\alpha_u(1) = \alpha_u(2) = 0$ for all u. Consequently, the asymptotic result of Corollary 1 can be transformed to a (feasible!) standard central limit theorem when the functions g and h are both even or odd (however, this is not a necessary condition). A similar assertion holds for the multivariate case presented in Theorem 2 (see Example 9).

Remark 2 *A natural extension of the generalised bipower variation $V(X,g,h)^n_t$ defined in (1.3) is the generalised multipower variation, which is given by*

$$V(X,g_1,\ldots,g_k)^n_t = \frac{1}{n}\sum_{i=1}^{[nt]-k+1}\prod_{j=1}^{k} g_j(\sqrt{n}\Delta^n_{i+j-1}X)\ , \qquad (2.15)$$

where $k \in I\!N$ is a fixed number. Clearly, a stable central limit theorem can be derived for the statistics of the type (2.15) (although we dispense with the exact exposition for the sake of notation). By similar arguments as presented above we can deduce that the limit process is mixed normal (and so the theory becomes feasible) when $2m$ of the k functions g_j are odd and the remaining $k-2m$ functions are even (for some $m \in I\!N$).

3 Some examples

In this Section we demonstrate some practical examples to illustrate the theoretical statements. In Examples 3–7 all processes are considered to be one-dimensional. Some more discussion on Examples 4 and 8 can be found in Barndorff-Nielsen, Graversen, Jacod & Shephard (2006).

Example 3 *(Toy Example)*
Assume that $g(x) = x$ and $h(x) = 1$. We immediately obtain

$$V(X,g,h)^n_t = \frac{1}{\sqrt{n}}(X_{[nt]/n} - X_0) \xrightarrow{P} V(X,g,h)_t = 0$$

for all t. Consequently, we have

$$\sqrt{n}(V(X,g,h)_t^n - V(X,g,h)_t) \xrightarrow{\mathcal{D}_{st}} \int_0^t a_u ds + \int_0^t \sigma_u dW_u \ ,$$

which is verified by Corollary 1.

Example 4 *(Realised Bipower Variation)*
Realised bipower variation, which is probably the most important subclass of (1.3), corresponds to the functions $g(x) = |x|^r$ and $h(x) = |x|^l$. Recalling Remark 1 and the convergence in probability in (2.4) we deduce the stable convergence ($r, l \geq 0$)

$$\sqrt{n}\Big(V(X,r,l)_t^n - \mu_r\mu_l \int_0^t |\sigma_u|^{r+l} du\Big) \xrightarrow{\mathcal{D}_{st}} U(r,l)_t \ ,$$

with

$$U(r,l)_t = \sqrt{\mu_{2r}\mu_{2l} + 2\mu_r\mu_l\mu_{r+l} - 3\mu_r^2\mu_l^2} \int_0^t |\sigma_u|^{r+l} \ dW'_u,$$

where W' is a 1-dimensional Brownian motion defined on an extension of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and is independent of the σ-field $\mathcal{F}$. Now we demonstrate how a feasible central limit theorem can be obtained.

Observe that the limit process $U(r,l)_t$ is mixed normal with conditional variance

$$\rho^2(r,l)_t = (\mu_{2r}\mu_{2l} + 2\mu_r\mu_l\mu_{r+l} - 3\mu_r^2\mu_l^2) \int_0^t |\sigma_u|^{2(r+l)} du.$$

By applying Theorem 1 we obtain the convergence

$$\rho^2(r,s)_t^n = \frac{\mu_{2r}\mu_{2l} + 2\mu_r\mu_l\mu_{r+l} - 3\mu_r^2\mu_l^2}{\mu_{2r}\mu_{2l}} V(X,2r,2l)_t^n \xrightarrow{P} \rho^2(r,l)_t. \tag{3.1}$$

Exploiting the properties of stable convergence (see Jacod & Shiryaev (2003)) we get

$$\frac{\sqrt{n}(V(X,r,s)_t^n - \mu_r\mu_l \int_0^t |\sigma_u|^{r+l} du)}{\rho(r,s)_t^n} \xrightarrow{\mathcal{D}} N(0,1). \tag{3.2}$$

This standard central limit theorem can be used to construct confidence regions. Note, however, that the properties of the weak convergence are not sufficient to deduce (3.2) from (3.1). This illustrates the importance of the concept of stable convergence.

Example 5 *(Realised Autocovariance)*
Realised autocovariance of lag h ($h \geq 1$) is defined by

$$\gamma(h)_t^n = \sum_{i=1}^{[nt]} \Delta_i^n X \Delta_{i+h}^n X \ ,$$

which corresponds to the case $g(x) = h(x) = x$ (note that the asymptotic theory does not depend on h when h is fixed). Realised autocovariances are used to construct the kernel-based estimator of integrated volatility derived by Barndorff-Nielsen, Hansen, Lunde & Shephard

(2006). It also appears implicitly in the two- and multiscale approach proposed by Zhang, Mykland & Ait-Sahalia (2005) and Zhang (2006). Both methods provide consistent estimates for the integrated volatility in the presence of (i.i.d) noise.

By Theorem 1 we immediately obtain

$$\gamma(h)_t^n \xrightarrow{P} 0.$$

Since g and h are both odd we have $\alpha_u(1) = \alpha_u(2) = 0$ for all u, $A_u = \sigma_u^4$, and we obtain the stable convergence

$$\sqrt{n}\gamma(h)_t^n \xrightarrow{\mathcal{D}_{st}} \int_0^t \sigma_u^2 dW_u' \ ,$$

where the limit is again mixed normal. The same arguments as presented in the previous Example yield the standard central limit theorem

$$\frac{\sqrt{n}\gamma(h)_t^n}{\sqrt{\frac{n}{3}\sum_{i=1}^{[nt]} |\Delta_i^n X|^4}} \xrightarrow{\mathcal{D}} N(0,1).$$

Example 6 *(The Cubic Power of Returns)*
Here we demonstrate Corollary 1 for $g(x) = x^3$ and $h(x) = 1$, i.e. for the statistic

$$\sqrt{n}\sum_{i=1}^{[nt]} (\Delta_i^n X)^3.$$

This quantity plays an important role in Jiang & Oomen (2006), who provide a test for finite activity jumps in the price process. Theorem 1 implies the convergence in probability

$$\sqrt{n}\sum_{i=1}^{[nt]} (\Delta_i^n X)^3 \xrightarrow{P} 0 \ ,$$

while

$$\alpha_u(1) = 3(a_u\sigma_u^2 + \sigma_u'\sigma_u^2) \ , \quad \alpha_u(2) = 3\sigma_u^3 \ , \quad \alpha_u(3) = \sqrt{6}|\sigma_u|^3.$$

Consequently, we obtain the stable convergence

$$n\sum_{i=1}^{[nt]} (\Delta_i^n X)^3 \xrightarrow{\mathcal{D}_{st}} 3\int_0^t (a_u\sigma_u^2 + \sigma_u'\sigma_u^2)du + 3\int_0^t \sigma_u^3 dW_u + \sqrt{6}\int_0^t |\sigma_u|^3 dW_u'.$$

Obviously, the latter result is not feasible. However, when $a_u = 0$ for all u and the volatility process σ is independent of the Brownian motion W (these are rather strong assumptions which, in particular, imply that $\sigma_u' = 0$ for all u), the limit process is again mixed normal and we deduce the standard central limit theorem

$$\frac{n\sum_{i=1}^{[nt]} (\Delta_i^n X)^3}{\sqrt{n^2\sum_{i=1}^{[nt]} |\Delta_i^n X|^6}} \xrightarrow{\mathcal{D}} N(0,1).$$

Example 7 *(Odd Power of Returns)*
We consider the case $g(x) = x^{2k+1}$ $(k \in I\!N)$, $h(x) = 1$, which is a generalisation of the latter Example. As above the convergence in probability

$$n^{\frac{2k-1}{2}} \sum_{i=1}^{[nt]} (\Delta_i^n X)^{2k+1} \xrightarrow{P} 0$$

holds. An application of Corollary 1 yields the stable convergence

$$n^k \sum_{i=1}^{[nt]} (\Delta_i^n X)^{2k+1} \xrightarrow{\mathcal{D}_{st}} \int_0^t \alpha_u(1) du + \int_0^t \alpha_u(2) dW_u + \int_0^t \alpha_u(3) dW'_u \ ,$$

where the processes $\alpha_u(1)$, $\alpha_u(2)$ and $\alpha_u(3)$ are given by

$$\begin{aligned} \alpha_u(1) &= \sigma_u^{2k} \Big(\frac{2k+1}{2} (\mu_{2k+2} - \mu_{2k}) \sigma'_u + (2k+1) \mu_{2k} a_u \Big) \ , \\ \alpha_u(2) &= \mu_{2k+2} \sigma_u^{2k+2} \ , \\ \alpha_u(3) &= \sqrt{\mu_{4k+2} - \mu_{2k+2}^2} \, |\sigma_u|^{2k+1} \ , \end{aligned}$$

respectively. Again, the stable central limit theorem is not feasible.

Now let us demonstrate some multivariate examples.

Example 8 *(Realised Covariation)*
The estimation of the covariation and related objects is probably the most important application of Theorem 2 in econometrics. Suppose that $d = d_1 = d_2 = d_3$, $g(x) = xx^T$, $h(x) = id_d$ and set

$$C = \sigma \sigma^T .$$

Theorem 1 implies the convergence

$$\sum_{i=1}^{[nt]} \Delta_i^n X \Delta_i^n X^T \xrightarrow{P} \int_0^t C_u du \ ,$$

where $\int_0^t C_u du$ is the covariation of the process X. Next, an application of Theorem 2 yields the stable convergence

$$\sqrt{n} \Big(\sum_{i=1}^{[nt]} \Delta_i^n X \Delta_i^n X^T - \int_0^t C_u du \Big) \xrightarrow{\mathcal{D}_{st}} \int_0^t A_u^{\frac{1}{2}} \, dW'_u \ ,$$

where the $d^2 \times d^2$-dimensional matrix A is defined componentwise by

$$A_u^{jk,j'k'} = \rho_{\sigma_u}(g^{jk} g^{j'k'}) - \rho_{\sigma_u}(g^{jk}) \rho_{\sigma_u}(g^{j'k'}).$$

A simple computation shows that

$$\begin{aligned}\rho_{\sigma_u}(g^{jk}) &= C_u^{jk} \ ,\\ \rho_{\sigma_u}(g^{jk}g^{j'k'}) &= C_u^{jk}C^{j'k'} + C^{jj'}C^{kk'} + C^{jk'}C^{kj'}.\end{aligned}$$

The latter formula is a direct consequence of the identity $Cov(U_1U_2, U_3U_4) = Cov(U_1, U_3)Cov(U_2, U_4) + Cov(U_1, U_4)Cov(U_2, U_3)$ for jointly normal variables $U_1, \ldots, U_4$. Now, we immediately obtain

$$A_u^{jk,j'k'} = C_u^{jj'}C_u^{kk'} + C_u^{jk'}C_u^{kj'}.$$

The central limit theorem above has first been published in Barndorff-Nielsen & Shephard (2004b) under a no-leverage assumption (i.e. when σ is independent of W) and was extended in Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006) to the general case. The result can be applied to derive the distribution theory for the *realised regression*

$$\frac{\sum_{i=1}^{[nt]} \Delta_i^n X^k \Delta_i^n X^l}{\sum_{i=1}^{[nt]} |\Delta_i^n X^k|^2} \xrightarrow{P} \frac{\int_0^t C_u^{kl} du}{\int_0^t C_u^{kk} du} \ ,$$

or for the *realised correlation*

$$\frac{\sum_{i=1}^{[nt]} \Delta_i^n X^k \Delta_i^n X^l}{\sqrt{\sum_{i=1}^{[nt]} |\Delta_i^n X^k|^2}\sqrt{\sum_{i=1}^{[nt]} |\Delta_i^n X^l|^2}} \xrightarrow{P} \frac{\int_0^t C_u^{kl} du}{\sqrt{\int_0^t C_u^{kk} du}\sqrt{\int_0^t C_u^{ll} du}}.$$

See Barndorff-Nielsen & Shephard (2004b) for more details on the asymptotic theory for the realised regression and realised correlation.

Example 9 *(Multivariate Realised Kernel)*
Here we consider the functions

$$g(x_1, x_2) = \begin{pmatrix} x_1 x_2 & 0 \\ 0 & x_1 \end{pmatrix} \ , \qquad h(x_1, x_2) = (1, x_2)^T.$$

The corresponding statistic $V(X, g, h)_t^n$ appears in the multivariate extension of the kernel-based approach (see Kinnebrock (2006) for a detailed study). Theorem 2 implies the central limit theorem

$$\sqrt{n} \begin{pmatrix} \sum_{i=1}^{[nt]} \Delta_i^n X^1 \Delta_i^n X^2 - \int_0^t C_u^{12} du \\ \sum_{i=1}^{[nt]} \Delta_i^n X^1 \Delta_{i+1}^n X^2 \end{pmatrix} \xrightarrow{\mathcal{D}_{st}} \int_0^t A_u^{\frac{1}{2}} \ dW'_u \ ,$$

where the matrix A_u is given by

$$A_u = \int_0^t \begin{pmatrix} (C_u^{12})^2 + C_u^{11}C_u^{22} & 0 \\ 0 & C_u^{11}C_u^{22} \end{pmatrix} du.$$

Note that the limit process is mixed normal although neither g nor h is an odd or even function.

4 Appendix

In the following we assume without loss of generality that the stochastic processes a, σ, a', σ' and v' are bounded (see Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006) for the justification of this assumption). Furthermore, we denote all constants that appear in the proof by C and we use the notation $Z^n \xrightarrow{P} Z$ if $\sup_{s\le t} |Z_t^n - Z_t| \xrightarrow{P} 0$ for all t.

We introduce the notation

$$\beta_i^n = \sqrt{n}\sigma_{\frac{i-1}{n}}\Delta_i^n W \ , \qquad \beta_i^{\prime n} = \sqrt{n}\sigma_{\frac{i-1}{n}}\Delta_{i+1}^n W. \tag{4.1}$$

Note that β_i^n (resp. $\beta_i^{\prime n}$) is an approximation of the quantity $\sqrt{n}\Delta_i^n X$ (resp. $\sqrt{n}\Delta_{i+1}^n X$). The assertion of Theorem 2 follows from the following three steps.

Step 1: When the functions g and h are continuous with at most polynomial growth then the stable convergence

$$U_t^n = \frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]}\Big(g(\beta_i^n)h(\beta_i^{\prime n}) - \rho_{\sigma_{\frac{i-1}{n}}}(g)\rho_{\sigma_{\frac{i-1}{n}}}(h)\Big) \xrightarrow{\mathcal{D}_{st}} \int_0^t \alpha_u(2)dW_u + \int_0^t \alpha_u(3)dW_u' \ , \tag{4.2}$$

where the processes $\alpha(2)$ and $\alpha(3)$ are defined in Theorem 2, holds.

Step 2: We consider the sequence of processes

$$U(g,h)_t^n = \frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]}\Big(g(\sqrt{n}\Delta_i^n X)h(\sqrt{n}\Delta_{i+1}^n X) - E[g(\sqrt{n}\Delta_i^n X)h(\sqrt{n}\Delta_{i+1}^n X)|\mathcal{F}_{\frac{i-1}{n}}]\Big). \tag{4.3}$$

If the functions g and h satisfy the same assumptions as in *Step 1*, it holds that

$$U(g,h)^n - U^n \xrightarrow{P} 0. \tag{4.4}$$

In fact, the convergence in (4.4) has been already proved in Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006), so we dispense with the exact exposition in this paper.

Step 3: In view of (4.2) and (4.4) the assertion of Theorem 2 follows from

$$\sqrt{n}(V(X,g,h)_t^n - V(X,g,h)_t) - U(g,h)_t^n \xrightarrow{P} \int_0^t \alpha_u(1)du. \tag{4.5}$$

In the following we will show that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]} E[g(\sqrt{n}\Delta_i^n X)h(\sqrt{n}\Delta_{i+1}^n X) - g(\beta_i^n)h(\beta_i^{\prime n})|\mathcal{F}_{\frac{i-1}{n}}] \xrightarrow{P} \int_0^t \alpha_u(1)du \ , \tag{4.6}$$

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]}\Big(\rho_{\sigma_{\frac{i-1}{n}}}(g)\rho_{\sigma_{\frac{i-1}{n}}}(h) - n\int_{\frac{i-1}{n}}^{\frac{i}{n}}\rho_{\sigma_u}(g)\rho_{\sigma_u}(h)du\Big) \xrightarrow{P} 0 \ , \tag{4.7}$$

which obviously imply (4.5).

Proof of Step 1: A straight forward calculation shows that

$$U_t^n = \sum_{i=2}^{[nt]+1} \zeta_i^n + o_p(1) , \tag{4.8}$$

with

$$\zeta_i^n = \frac{1}{\sqrt{n}}\Big(g(\beta_{i-1}^n)\{h(\beta_{i-1}^{\prime n}) - \rho_{\sigma_{\frac{i-2}{n}}}(h)\} + \rho_{\sigma_{\frac{i-1}{n}}}(h)\{g(\beta_i^{\prime n}) - \rho_{\sigma_{\frac{i-1}{n}}}(g)\}\Big). \tag{4.9}$$

We clearly have

$$E[\zeta_i^n|\mathcal{F}_{\frac{i-1}{n}}] = 0 ,$$

while ($1 \le j, j' \le d_1$, $1 \le k, k' \le d_3$)

$$\begin{aligned} &E[\zeta_i^{n,jk}\zeta_i^{n,j'k'}|\mathcal{F}_{\frac{i-1}{n}}] \\ &= \frac{1}{n}\sum_{l,l'=1}^{d_2}\Big(g(\beta_{i-1}^n)^{jl}g(\beta_{i-1}^n)^{j'l'}\ \{\rho_{\sigma_{\frac{i-2}{n}}}(h^{lk}h^{l'k'}) - \rho_{\sigma_{\frac{i-2}{n}}}(h^{lk})\rho_{\sigma_{\frac{i-2}{n}}}(h^{l'k'})\} \\ &\quad +g(\beta_{i-1}^n)^{jl}\ \rho_{\sigma_{\frac{i-1}{n}}}(h^{l'k'})\ \{\rho_{i-2,i-1}^n(g^{j'l'}, h^{lk}) - \rho_{\sigma_{\frac{i-2}{n}}}(h^{lk})\rho_{\sigma_{\frac{i-1}{n}}}(g^{j'l'})\} \\ &\quad +g(\beta_{i-1}^n)^{j'l'}\ \rho_{\sigma_{\frac{i-1}{n}}}(h^{lk})\ \{\rho_{i-2,i-1}^n(g^{jl}, h^{l'k'}) - \rho_{\sigma_{\frac{i-2}{n}}}(h^{l'k'})\rho_{\sigma_{\frac{i-1}{n}}}(g^{jl})\} \\ &\quad +\rho_{\sigma_{\frac{i-1}{n}}}(h^{l'k'})\rho_{\sigma_{\frac{i-1}{n}}}(h^{lk})\ \{\rho_{\sigma_{\frac{i-1}{n}}}(g^{jl}g^{j'l'}) - \rho_{\sigma_{\frac{i-1}{n}}}(g^{jl})\rho_{\sigma_{\frac{i-1}{n}}}(g^{j'l'})\}\Big) , \end{aligned}$$

with $\rho_{i-2,i-1}^n(g,h) = \int g(\sigma_{\frac{i-1}{n}}x)h(\sigma_{\frac{i-2}{n}}x)\rho(dx)$, where ρ is the $N(0, I_{d'})$ law. This implies

$$\sum_{i=2}^{[nt]+1} E[\zeta_i^{n,jk}\zeta_i^{n,j'k'}|\mathcal{F}_{\frac{i-1}{n}}] \xrightarrow{P} \int_0^t A_u^{jk,j'k'}du. \tag{4.10}$$

Next, for $1 \le j \le d_1$, $1 \le k \le d_3$ and $1 \le r \le d'$ we obtain

$$E[\zeta_i^{n,jk}\Delta_i^n W^r|\mathcal{F}_{\frac{i-1}{n}}] = \frac{1}{n}\sum_{l=1}^{d_2}\Big(\rho_{\sigma_{\frac{i-2}{n}}}(g^{jl})\rho^{(k)}_{\sigma_{\frac{i-2}{n}}}(h^{lk}) + \rho_{\sigma_{\frac{i-1}{n}}}(h^{lk})\rho^{(k)}_{\sigma_{\frac{i-1}{n}}}(g^{jl})\Big) ,$$

from which we deduce

$$\sum_{i=2}^{[nt]+1} E[\zeta_i^{n,jk}\Delta_i^n W^r|\mathcal{F}_{\frac{i-1}{n}}] \xrightarrow{P} \int_0^t \Big(\rho_{\sigma_u}(g^{jl})\rho^{(k)}_{\sigma_u}(h^{lk}) + \rho_{\sigma_u}(h^{lk})\rho^{(k)}_{\sigma_u}(g^{jl})\Big)du. \tag{4.11}$$

Finally, let N be any bounded martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$ which is orthogonal to W (i.e. the covariation $< N, W >_t = 0$ a.s.). Then we deduce

$$E[\zeta_i^{n,jk}\Delta_i^n N|\mathcal{F}_{\frac{i-1}{n}}] = 0 , \tag{4.12}$$

which has been already shown in Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006). Now, the assertion of *Step 1* follows from (4.10), (4.11), (4.12) and Theorem IX.7.28 in Jacod & Shiryaev (2003). □

Proof of Step 3: The proof of (4.7) can be found in Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006), so we concentrate on proving (4.6).

First, note that the identity

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]} E[g(\sqrt{n}\Delta_i^n X)h(\sqrt{n}\Delta_{i+1}^n X) - g(\beta_i^n)h(\beta_i^{\prime n})|\mathcal{F}_{\frac{i-1}{n}}] = \sum_{i=1}^{[nt]} \mu_i^n ,$$

with

$$\mu_i^n = \frac{1}{\sqrt{n}}\; E\Big[g(\sqrt{n}\Delta_i^n X)\Big(h(\sqrt{n}\Delta_{i+1}^n X) - h(\beta_i^{\prime n})\Big) + \Big(g(\sqrt{n}\Delta_i^n X) - g(\beta_i^n)\Big)h(\beta_i^{\prime n})|\mathcal{F}_{\frac{i-1}{n}}\Big] ,$$

holds. Under assumption (V) we introduce the following $\mathbb{R}^d$-valued random variables

$$\begin{aligned} \zeta(1)_i^n &= \sqrt{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} (a_u - a_{\frac{i-1}{n}})du + \sqrt{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\left(\int_{\frac{i-1}{n}}^{u} a_s' ds \right. \\ &\qquad \left. + \int_{\frac{i-1}{n}}^{u} (\sigma_s' - \sigma'_{\frac{i-1}{n}})dW_s + \int_{\frac{i-1}{n}}^{u} (v_s' - v'_{\frac{i-1}{n}})dV_s\right) dW_u , \\ \zeta(1)_i^{\prime n} &= \sqrt{n}\left(\int_{\frac{i-1}{n}}^{\frac{i}{n}} a_s' ds + \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(\sigma_s' - \sigma'_{\frac{i-1}{n}}\right) dW_s \right. \\ &\qquad \left. + \int_{\frac{i-1}{n}}^{\frac{i}{n}} (v_s' - v'_{\frac{i-1}{n}})dV_s\right) \Delta_{i+1}^n W , \\ \zeta(2)_i^n &= \sqrt{n}\left(\frac{1}{n}\; a_{\frac{i-1}{n}} + \sigma'_{\frac{i-1}{n}}\int_{\frac{i-1}{n}}^{\frac{i}{n}} (W_u - W_{\frac{i-1}{n}})dW_u \right. \\ &\qquad \left. + v'_{\frac{i-1}{n}}\int_{\frac{i-1}{n}}^{\frac{i}{n}} (V_u - V_{\frac{i-1}{n}})dW_u\right) , \\ \zeta(2)_i^{\prime n} &= \sqrt{n}\; \left(\sigma'_{\frac{i-1}{n}}\Delta_i^n W + v'_{\frac{i-1}{n}}\Delta_i^n V\right) \Delta_{i+1}^n W. \end{aligned} \tag{4.13}$$

A simple computation shows that

$$\begin{aligned} \sqrt{n}\Delta_i^n X - \beta_i^n &= \zeta(1)_i^n + \zeta(2)_i^n , \\ \sqrt{n}\Delta_{i+1}^n X - \beta_i^{\prime n} &= \zeta(1)_{i+1}^n + \zeta(1)_i^{\prime n} + \zeta(2)_i^{\prime n} + \zeta(2)_{i+1}^n. \end{aligned} \tag{4.14}$$

The Taylor expansion yields the representation

$$\mu_i^n = \sum_{k=1}^{6} \mu_i^n(k) ,$$

where the quantities $\mu_i^n(k) \in I\!R^{d_1 \times d_3}$ $(1 \le k \le 6)$ are given by

$$\mu_i^n(1) = \frac{1}{\sqrt{n}}\ E\Big[g(\sqrt{n}\Delta_i^n X)\nabla h(\beta_i'^n)\zeta(2)_i'^n|\mathcal{F}_{\frac{i-1}{n}}\Big]\ ,$$

$$\mu_i^n(2) = \frac{1}{\sqrt{n}}\ E\Big[g(\sqrt{n}\Delta_i^n X)\nabla h(\beta_i'^n)\zeta(2)_{i+1}^n|\mathcal{F}_{\frac{i-1}{n}}\Big]\ ,$$

$$\mu_i^n(3) = \frac{1}{\sqrt{n}}\ E\Big[\nabla g(\beta_i^n)\zeta(2)_i^n h(\beta_i'^n)|\mathcal{F}_{\frac{i-1}{n}}\Big]\ ,$$

$$\mu_i^n(4) = \frac{1}{\sqrt{n}}\ E\Big[g(\sqrt{n}\Delta_i^n X)\nabla h(\beta_i'^n)(\zeta(1)_{i+1}^n + \zeta(1)_i'^n) + \nabla g(\beta_i^n)\zeta(1)_i^n h(\beta_i'^n)|\mathcal{F}_{\frac{i-1}{n}}\Big]\ ,$$

$$\mu_i^n(5) = \frac{1}{\sqrt{n}}\ E\Big[g(\sqrt{n}\Delta_i^n X)\Big(\nabla h(\bar\gamma_i'^n) - \nabla h(\beta_i'^n)\Big)\Big(\sqrt{n}\Delta_{i+1}^n X - \beta_i'^n\Big)|\mathcal{F}_{\frac{i-1}{n}}\Big]\ ,$$

$$\mu_i^n(6) = \frac{1}{\sqrt{n}}\ E\Big[\Big(\nabla g(\bar\gamma_i^n) - \nabla g(\beta_i^n)\Big)\Big(\sqrt{n}\Delta_i^n X - \beta_i^n\Big)h(\beta_i'^n)|\mathcal{F}_{\frac{i-1}{n}}\Big]\ ,$$

for some random d-dimensional variables $\bar\gamma_i^n$, $\bar\gamma_i'^n$ that satisfy

$$|\bar\gamma_i^n - \beta_i^n| \le |\sqrt{n}\Delta_i^n X - \beta_i^n|\ , \qquad |\bar\gamma_i'^n - \beta_i'^n| \le |\sqrt{n}\Delta_{i+1}^n X - \beta_i'^n|.$$

By the arguments presented in Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006) we obtain

$$\sum_{i=1}^{[nt]} \mu_i^n(k) \xrightarrow{P} 0 \tag{4.15}$$

for $k = 4, 5, 6$. A straight forward estimation gives

$$\sum_{i=1}^{[nt]} \mu_i^n(1) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} E\Big[g(\beta_i^n)\nabla h(\beta_i'^n)\zeta(2)_i'^n|\mathcal{F}_{\frac{i-1}{n}}\Big] + o_p(1). \tag{4.16}$$

Next, a tedious but simple calculation (and (4.16)) shows that $(1 \le j \le d_1,\ 1 \le k \le d_3)$

$$\begin{aligned}
\sum_{i=1}^{[nt]} \mu_i^n(1)^{jk} \quad &= \quad \frac{1}{n}\sum_{i=1}^{[nt]}\sum_{l=1}^{d_2}\sum_{r=1}^{d}\sum_{s=1}^{d'}\sum_{p=1}^{d'} \sigma'^{rsp}_{\frac{i-1}{n}}\ \rho^{(s)}_{\sigma_{\frac{i-1}{n}}}(g^{jl})\ \rho^{(s)}_{\sigma_{\frac{i-1}{n}}}(\frac{\partial h^{lk}}{\partial r}) + o_p(1) \\
&\xrightarrow{P} \quad \int_0^t \sum_{l=1}^{d_2}\sum_{r=1}^{d}\sum_{s=1}^{d'}\sum_{p=1}^{d'} \sigma'^{rsp}_u\ \rho^{(s)}_{\sigma_u}(g^{jl})\ \rho^{(s)}_{\sigma_u}(\frac{\partial h^{lk}}{\partial r})\ du.
\end{aligned} \tag{4.17}$$

Similarly, we obtain

$$\begin{aligned}
\sum_{i=1}^{[nt]} \mu_i^n(2)^{jk} \quad &= \quad \frac{1}{n}\sum_{i=1}^{[nt]}\Big(\sum_{l=1}^{d_2}\sum_{r=1}^{d}\sum_{s=1}^{d'}\sum_{p=1}^{d'} \sigma'^{rsp}_{\frac{i-1}{n}}\rho_{\sigma_{\frac{i-1}{n}}}(g^{jl})\tilde\rho^{(sp)}_{\sigma_{\frac{i-1}{n}}}(\frac{\partial h^{lk}}{\partial r}) \\
&+ \quad \sum_{l=1}^{d_2}\sum_{r=1}^{d} a^r_{\sigma_{\frac{i-1}{n}}}\rho_{\sigma_{\frac{i-1}{n}}}(g^{jl})\rho_{\sigma_{\frac{i-1}{n}}}(\frac{\partial h^{lk}}{\partial r})\Big) + o_p(1)
\end{aligned}$$

$$\xrightarrow{P} \int_0^t \Big(\sum_{l=1}^{d_2} \sum_{r=1}^{d} \sum_{s=1}^{d'} \sum_{p=1}^{d'} \sigma_u'^{rsp} \rho_{\sigma_u}(g^{jl}) \tilde{\rho}_{\sigma_u}^{(sp)}(\frac{\partial h^{lk}}{\partial r}) + \sum_{l=1}^{d_2} \sum_{r=1}^{d} a_{\sigma_u}^r \rho_{\sigma_u}(g^{jl}) \rho_{\sigma_u}(\frac{\partial h^{lk}}{\partial r}) \Big)\, du \quad (4.18)$$

and

$$\sum_{i=1}^{[nt]} \mu_i^n(3)^{jk} = \frac{1}{n} \sum_{i=1}^{[nt]} \sum_{l=1}^{d_2} \sum_{r=1}^{d} \sum_{s=1}^{d'} \sum_{p=1}^{d'} \Big(a_{\frac{i-1}{n}}^r \rho_{\sigma_{\frac{i-1}{n}}}(\frac{\partial g^{jl}}{\partial r}) \rho_{\sigma_{\frac{i-1}{n}}}(h^{lk})$$

$$+\sigma_{\frac{i-1}{n}}'^{rsp} \rho_{\sigma_{\frac{i-1}{n}}}(h^{lk}) \tilde{\rho}_{\sigma_{\frac{i-1}{n}}}^{(sp)}(\frac{\partial g^{jl}}{\partial r}) \Big) \quad (4.19)$$

$$\xrightarrow{P} \int_0^t \sum_{l=1}^{d_2} \sum_{r=1}^{d} \sum_{s=1}^{d'} \sum_{p=1}^{d'} \Big(a_u^r \rho_{\sigma_u}(\frac{\partial g^{jl}}{\partial r}) \rho_{\sigma_u}(h^{lk}) + \sigma_u'^{rsp} \rho_{\sigma_u}(h^{lk}) \tilde{\rho}_{\sigma_u}^{(sp)}(\frac{\partial g^{jl}}{\partial r}) \Big)\, du.$$

By combining (4.15), (4.17), (4.18) and (4.19) we readily deduce the convergence of (4.6), which completes the proof of *Step 3*. □

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