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Gibbs measures

(1) Mean Field models, entropy, phase transitions,
disordered mean field models

(2) Metastates in finite-type mean-field models: visibility, invisibility,
and random restoration of symmetry

(3) Gibbs measures on the lattice and on trees
Dobrushin uniqueness criterion
Phase transitions

(4) Spin Dynamics, non-Gibbsian measures and multiple histories

Main example, the Ising model in mean field

***Definition* 1.** *Take as probability space* $\Omega_N = \{-1, 1\}^N$*. The Gibbs distribution of the Curie-Weiss model at inverse temperature* β *is given by*

$$\mu_{\beta,N}(\sigma_1, \ldots, \sigma_N) = \frac{1}{Z_{\beta,N}} \exp\left(\frac{\beta}{2N} \sum_{1\le i,j\le N} \sigma_i\sigma_j\right)$$

$$= \frac{1}{Z_{\beta,N}} \exp\left(N\frac{\beta}{2}\left(\frac{1}{N}\sum_{1\le i\le N}\sigma_i\right)^2\right)$$

***Exercise* 2.** *Is* $\mu_{\beta,N}$ *a compatible family of measures?*

As usual we want to understand a large system. Let us put on a short-sighted view, and consider only the marginal distribution on a fixed finite number k of components, when N gets large. So a prototypical question is:

***Question* 3.** *What is* $\mu_{\beta,N}(\sigma_1, \ldots, \sigma_k)$ *for large* N*?*

This has an answer:

***THEOREM* 4.** $\lim_{N\uparrow\infty}\mu_{\beta,N}(\sigma_1,\ldots,\sigma_k) = \left(\frac{1}{2}\right)^k$ *iff* $\beta \leq 1$.

For $\beta > 1$ *one has* $\lim_{N\uparrow\infty}\mu_{\beta,N}(\sigma_1,\ldots,\sigma_k) = \frac{1}{2}\left[\prod_{i=1}^k \frac{e^{\beta m^*\sigma_i}}{2\cosh(\beta m*)} + \prod_{i=1}^k \frac{e^{-\beta m^*\sigma_i}}{2\cosh(\beta m*)}\right]$
where m^* *is the biggest solution to the equation* $m^* = \tanh\beta m^*$.

This signifies so-called ferromagnetic order, or symmetry breaking.

***Exercise* 5.** *Which symmetries does the measure* $\mu_{\beta,N}$ *possess?*

We can extend this definition of a mean-field model to the following set-up.

***Definition* 6.** *Suppose that* $F : [-1, 1] \mapsto \mathbb{R}$ *is a continuous function, bounded below. Then the Curie-Weiss model with Hamiltonian* $NF(m)$ *is given by*

$$\mu_{F,N}(\sigma_1, \dots, \sigma_N) = \frac{1}{Z_{F,N}} \exp\left(-NF\left(\frac{1}{N}\sum_{1\le i\le N} \sigma_i\right)\right)$$

Same question. Note again that the measure is *exchangeable*, that is permutation invariant, in the following sense.

***Definition* 7.** *A probability measure* $\mu(\sigma_1, \dots, \sigma_N)$ *is called exchangeable if*

$$\mu(\sigma_1, \dots, \sigma_N) = \mu(\sigma_{\pi(1)}, \dots, \sigma_{\pi(N)})$$

for all permutations π*.*

Definition 8. *Let π, α be two probability measures on the finite state space E. Then*

$$S(\pi|\alpha) = \sum_{x \in E} \pi(x) \log \frac{\pi(x)}{\alpha(x)}$$

is called the relative entropy.

Note that we can rewrite in the form

$$\begin{aligned} S(\pi|\alpha) &= \sum_{x \in E} \alpha(x) \left[\frac{\pi(x)}{\alpha(x)} \log \frac{\pi(x)}{\alpha(x)} - \frac{\pi(x)}{\alpha(x)} + 1 \right] \\ &=: \sum_{x \in E} \alpha(x) \psi\left(\frac{\pi(x)}{\alpha(x)} \right) \end{aligned}$$

Looking at the graph of the function $\psi(x) = x \log x - x + 1$ we see that S is non-negative and zero if and only if $\pi(x) = \alpha(x)$ for all x.

***THEOREM* 9.** *(Georgii Theorem 3.A3). For a strictly postive stochastic matrix M and the unique probability vector α such that $\alpha M = \alpha$ we have*

$$\lim_{n\uparrow\infty} M^n(x,y) = \alpha(y)$$

Proof: The idea of the proof is to show that, for all $\pi \neq \alpha$ the relative entropy strictly decreases under application of M, and use this to deduce that πM^n actually converges to α.
In order to show $S(\pi M|\alpha) < S(\alpha)$ we use Jensen's inequality.

A convex function and a probability measure can be made to appear in the representation of the relative entropy in the following way:

$$
\begin{aligned}
S(\pi M|\alpha) &= \sum_x \alpha(x)\psi\Big(\frac{\pi M(x)}{\alpha(x)}\Big) \\
&= \sum_x \alpha(x)\psi\Big(\sum_y \pi(y)M(y,x)\frac{1}{\alpha(x)}\Big) \\
&= \sum_x \alpha(x)\psi\Big(\sum_y \alpha(y)M(y,x)\frac{1}{\alpha(x)}\frac{\pi(y)}{\alpha(y)}\Big)
\end{aligned}
$$

Note that $\sum_y \alpha(y)M(y,x)\frac{1}{\alpha(x)} = 1$ because of invariance of α.
In fact, we note in passing that $M'(x,y) := \alpha(y)M(y,x)\frac{1}{\alpha(x)}$ is the transition matrix for the *time-reversed chain*, i.e. $M'(x,y) = \frac{\mu_M(\sigma(0)=y,\sigma(1)=x)}{\mu_M(\sigma(1)=x)}$.

So, with Jensen we have from here

$$
\begin{aligned}
&\sum_x \alpha(x)\psi\Big(\sum_y \alpha(y)M(y,x)\frac{1}{\alpha(x)}\frac{\pi(y)}{\alpha(y)}\Big)\\
&< \sum_x \alpha(x)\sum_y \alpha(y)M(y,x)\frac{1}{\alpha(x)}\psi\Big(\frac{\pi(y)}{\alpha(y)}\Big)\\
&= \sum_y \alpha(y)\psi\Big(\frac{\pi(y)}{\alpha(y)}\Big) = S(\pi|\alpha)
\end{aligned}
$$

αM^n can not stay at a finite distance to α, by compactness:

Consider the set of probability vectors $K_\varepsilon = \{\pi, \sum_y |\pi(y) - \alpha(y)| \geq \varepsilon\}$ which keep a finite distance ε to α. In order to show convergence, we must show that αM^n exits K_ε after a finite time. But suppose it spends an infinite time in K_ε. Put $\delta = \inf_{\pi \in K_\varepsilon}(S(\pi|\alpha) - S(\pi M|\alpha))$. Then $S(\pi|\alpha) \geq \delta\infty = \infty$ to begin with which is a contradiction, because every probability vector π has a finite relative entropy w.r.t. α, if $\alpha(x) > 0$ for all $x \in E$. $\square$

relative entropy= Lyapunov function

A sophisticated extension of the argument can be used to prove a relation between the *invariant measures of a Markov dynamics in infinite volume* and the notion of *Gibbs measures in the infinite volume* (Stroock, Fritz, . . .)

***Exercise* 10.** *Prove that $S(\mu|\nu)$ is a convex function jointly in the pair (μ, ν).*

***Exercise* 11.** *Take a product space $E \times E'$ with two joint measures K, K'. Denote by K_1, K_1' the marginals on the first coordinate. Prove or disprove $S(K_1, K_1') \leq S(K, K')$.*

***Exercise* 12.** *Is the symmetrized entropy $D_s(\mu, \nu) := S(\mu|\nu) + S(\nu|\mu)$ a metric? Hint: Check the Bernoulli-case, write $\bar{D}_s(p, q) := D_s(\mu_p, \mu_q)$ where μ_p is the measure with $\mu_p(+1) = p, \mu_p(-1) = 1 - p$.*

***Definition* 13.** *Let α be a probability measures on the finite state space E. Then*

$$H(\alpha) = -\sum_{x \in E} \alpha(x) \log \alpha(x) \geq 0$$

is called the entropy of α, where it is assumed that $0 \log 0 := 0$.

Entropy can be seen as a measure of uncertainty of α.

Indeed, if $\alpha(x)$ is a Dirac measure (i.e. $\alpha(x) = 1_{x=x_0}$ for a particular value $x_0 \in E$, it vanishes. If $\alpha(x) = \alpha_0(x) := \frac{1}{|E|}$ is the equidistribution, then we have

$H(\alpha_0) = \log |E|$

$S(\pi|\alpha_0) = -H(\pi) + \log |E|$.

Let us take ν a probability measure on the finite state space E, Entropy $H(\nu)$
Boltzmann: *Entropy is, asympototically for large* n*, equal to* $\frac{1}{n}\log$ *of the number of microstates associated to a given macrostate.*

The role of the macrostate is ν. The microstates are the elements in the set

$$\Omega(\nu) := \{\omega \in E^n | L_n(\omega) = \nu\} = L_n^{-1}(\nu)$$

where $L_n(\omega) = \frac{1}{n}\sum_{i=1}^{n} \delta_{\omega_i}$ is the empirical distribution associated to ω.
For $n\nu(x) =: k(x)$ to be integer-valued, have multinomial coefficient

$$|\Omega(\nu)| = \frac{n!}{\prod_{x\in E} k(x)!}$$

This is the number of microstates ω compatible with ν.

***Lemma* 14.** *For a product measure* μ*, the* μ*-probability to draw a particular word* $\omega \in \Omega(\nu)$ *that is compatible with* ν *is*

$$\mu(\omega) = \exp\Big(-n(H(\nu) + S(\nu|\mu))\Big)$$

Proof:

$$
\begin{aligned}
&\mu(\omega) = \mu(\omega_1)\ldots\mu(\omega_n) \\
&= \prod_{x\in E} \mu(x)^{k(x)} = \exp \sum_x k(x) \log \mu(x) \\
&= \exp n \sum_x \nu(x) \log \mu(x) \\
&= \exp\Big(n(\sum_x \nu(x) \log \frac{\mu(x)}{\nu(x)} + \sum_x \nu(x) \log \nu(x))\Big)
\end{aligned}
$$

$\square$

Connection to Boltzmann: entropy is one over n times $\log$ of the number of microstates corresponding to one macrostate, up to a correction factor which is polynomial in the size of the system:

***Lemma* 15.** *If $n\nu(x)$ is integer-valued,*

$$(n+1)^{-|E|}e^{nH(\nu)} \leq |\Omega(\nu)| \leq e^{nH(\nu)}$$

Proof: A possible proof follows from Stirling's formula. In our proof we follow Dembo Zeitouni book, which uses no computation.

The upper bound: Take an $\omega \in \Omega(\nu)$

$$1 \geq \nu(\Omega(\nu)) = |\Omega(\nu)|\nu(\omega) = |\Omega(\nu)|e^{-n(H(\nu))+H(\nu|\nu)} = |\Omega(\nu)|e^{-nH(\nu)}$$

The lower bound: This will follow by the intuitive statement that

$$\nu(\Omega(\nu')) \leq \nu(\Omega(\nu))$$

Indeed, it should be less probable to see a ν'-like state under ν than to see a ν-like state. If we assume this we conclude that

$$1 = \sum_{\nu'} \nu(\Omega(\nu')) \leq (n+1)^{|E|}\nu(\Omega(\nu)) = (n+1)^{|E|}|\Omega(\nu)|e^{-nH(\nu)}$$

since the number of empirical measures with alphabet E of size n is bounded by $(n+1)^{|E|}$.

Assume for simplicity that all $\nu'(x)$ are strictly positive and put $k'(x) = n\nu(x)$. Then,

$$\frac{\nu(\Omega(\nu))}{\nu(\Omega(\nu'))} = \frac{|\Omega(\nu)| \prod_x \nu(x)^{k(x)}}{|\Omega(\nu')| \prod_x \nu(x)^{k'(x)}}$$
$$= \prod_x \frac{1}{k(x)!} \frac{k'(x)!}{1} \nu(x)^{k(x)-k'(x)}$$

Use the fact that $\frac{k!'}{k!} \geq k^{k'-k}$. This gives the lower bound

$$\prod_x (n\nu(x))^{k(x)-k'(x)} \nu(x)^{k(x)-k'(x)}$$
$$= n^{\sum_x (k(x)-k'(x))} = 1$$

$\square$

We can now combine the results to see the meaning of the relative entropy.

***THEOREM* 16.** *Discrete version of Sanov's theorem. If $n\nu(x)$ is integer valued we have the upper and lower large deviation bounds*

$$(n+1)^{-|E|}e^{-nS(\mu|\nu)} \leq \mu(\Omega(\nu)) = \mu(\omega : L_n(\omega) = \nu) \leq e^{-nS(\mu|\nu)}$$

Proof: Pick an $\omega_\nu \in \Omega(\nu)$ and write

$$\mu(\Omega(\nu)) = \mu(\omega_\nu)|\Omega(\nu)| = e^{-n(H(\nu)+S(\mu|\nu))}|\Omega(\nu)|$$

and use the expression on the size of $\Omega(\nu)$ in terms of entropy. $\square$

As a consequence of the last finite volume statement we get Sanov's Theorem in the following form. It is an example of the formulation of a large deviation principle.

***THEOREM* 17.** *Write* $P = \{p = (p(x))_{x\in E}, p(x) \geq 0, \sum_{x\in E} p(x) = 1\}$ *for the simplex of probability vectors, viewed a subset in* $\mathbb{R}^{|E|+1}$*.*
Suppose that $A \subset P$*, and denote by* A^o *its interior in* $\mathcal{P}(E)$*. Then*

$$
\begin{aligned}
&-\inf_{\nu\in A^o} S(\nu|\mu) \leq \liminf_{n\uparrow\infty} \frac{1}{n} \log \mu(L_n \in A) \\
&\leq \limsup_{n\uparrow\infty} \frac{1}{n} \log \mu(L_n \in A) \leq -\inf_{\nu\in A} S(\nu|\mu)
\end{aligned}
$$

Proof. See notes.

The interior on the l.h.s. is necessary, as examples of $A = \{\nu_0\}$ show.

Suppose that $F:\mathcal{P}(E)\mapsto\mathbb{R}$ is *a* continuous function, bounded below, E finite.

***Definition* 18.** *The mean field model with Hamiltonian $NF(\nu)$ and a priori measure $\alpha\in\mathcal{P}(E)$ is given by*

$$\mu_{F,N}(\sigma_1,\ldots,\sigma_N)=\frac{1}{Z_{F,N}}\exp\left(-NF\left(L_N(\sigma)\right)\right)\prod_{i=1}^{n}\alpha(\sigma_i)$$

We compute

$$\mu_{F,N}(L_N(\sigma)=\nu)=\frac{\alpha^n(\Omega(\nu))e^{-nF(\nu)}}{\sum_{\tilde{\nu}\in M_n}\alpha^n(\Omega(\tilde{\nu}))e^{-nF(\tilde{\nu})}}$$

Using the finite volume discrete Sanov theorem we get the bounds

$$\mu_{F,N}(L_N(\sigma)=\nu)\leq(N+1)^{|E|}\frac{e^{-N(S(\nu|\alpha)+F(\nu))}}{\sum_{\tilde{\nu}\in M_n}e^{-N(S(\tilde{\nu}|\alpha)+F(\tilde{\nu}))}}$$

and

$$\mu_{F,N}(L_N(\sigma)=\nu)\geq(N+1)^{-|E|}\frac{e^{-N(S(\nu|\alpha)+F(\nu))}}{\sum_{\tilde{\nu}\in M_n}e^{-N(S(\tilde{\nu}|\alpha)+F(\tilde{\nu}))}}$$

So, $\mu_{F,N}(L_N(\sigma) = \cdot)$ concentrates exponentially fast around the minimizers of $\nu \mapsto S(\nu|\alpha) + F(\nu)$, given α.
As a consequence we get immediately the behavior on the level of the law of large numbers.
We write $\|\pi - \pi'\| = \frac{1}{2}\sum_{a \in E} |\pi(a) - \pi'(a)| = \sup_{A \subset E} |\pi(A) - \pi'(A)|$.

***THEOREM* 19.** *(Law of large numbers.) If $\nu \mapsto S(\nu|\alpha) + F(\nu)$ has a unique minimizer ν^* in $\mathcal{P}(E)$ then*

$$\lim_{N\uparrow\infty} \mu_{F,N}(\|L_N - \nu^*\| \geq \varepsilon) = 0.$$

Proof:

$$\mu_{F,N}(\|L_N - \nu^*\| \geq \varepsilon) \leq (N+1)^{2|E|} \frac{e^{-N \inf_{\nu \in M_n : \|\nu - \nu^*\| \geq \varepsilon}(S(\nu|\alpha)+F(\nu))}}{e^{-N \inf_{\nu \in M_n}(S(\nu|\alpha)+F(\nu))}}$$

Now, because of the continuity of $\nu \mapsto S(\nu|\alpha) + F(\nu)$ we have

$$\lim_{n \uparrow \infty} \inf_{\nu \in M_n} (S(\nu|\alpha) + F(\nu)) = S(\nu^*|\alpha) + F(\nu^*)$$

We also have

$$\begin{aligned}
&\inf_{\nu \in M_n : \|\nu - \nu^*\| \geq \varepsilon} (S(\nu|\alpha) + F(\nu)) - (S(\nu^*|\alpha) + F(\nu^*)) \\
&\geq \inf_{\nu \in M : \|\nu - \nu^*\| \geq \varepsilon} (S(\nu|\alpha) + F(\nu)) - (S(\nu^*|\alpha) + F(\nu^*)) \\
&\geq \delta = \delta(\varepsilon) > 0
\end{aligned}$$

since $\{\nu \in M : \|\nu - \nu^*\| \geq \varepsilon\}$ is compact $\square$

An analogy of the previous LLN is as follows.

THEOREM 20. *(”Law of large numbers”.) Denote by $M^* \subset \mathcal{P}(E)$ the set of minimizers of the map $\nu \mapsto S(\nu|\alpha) + F(\nu)$. Suppose that M^* is a finite set.*

1. *Then*

$$\lim_{N\uparrow\infty} \mu_{F,N}(\inf_{\nu^*\in M^*} \|L_N - \nu^*\| \geq \varepsilon) = 0.$$

2. *If there is moreover a group T acting as transformations $\tau : \mathcal{P}(E) \mapsto \mathcal{P}(E)$ for $\tau \in T$ which preserves the rate function, $S(\nu|\alpha) + F(\nu) = S(\tau\nu|\alpha) + F(\tau\nu)$, and $M^* = \{\tau\nu^*, \tau \in T\}$ then we have the symmetric expression*

$$\lim_{N\uparrow\infty} \mu_{F,N}(G(L_N)) = \frac{1}{|M^*|} \sum_{\nu^*\in M^*} G(\delta_{\nu^*})$$

for all bounded continuous G on $\mathcal{P}(E)$.

THEOREM 21. *For the standard Curie Weiss Ising model we have*

1.
$$\lim_{N\uparrow\infty} \mu_{\beta,N}(\|L_N - \nu_0\| \geq \varepsilon) = 0$$
iff $\beta \leq 1$.

2. *For* $\beta > 1$ *one has*
$$\lim_{N\uparrow\infty} \mu_{\beta,N}(\|L_N - \nu_{sm^*}\| \leq \varepsilon) = \frac{1}{2}$$
for $s = \pm 1$ *where* m^* *is the biggest solution to the equation* $m^* = \tanh \beta m^*$,
$$\nu_m = \frac{1+m}{2}\delta_+ + \frac{1-m}{2}\delta_-$$
.

Proof: We have

$$H_\beta(\nu_m) + S(\nu_m|m_0) = -\frac{\beta m^2}{2} + \frac{1+m}{2}\log(1+m) + \frac{1-m}{2}\log(1-m)$$

Solving the minimization problem over $m \in (-1, 1)$ leads to the answer. $\square$

For a positive integer q, the Gibbs measure $\mu^N_{q,\beta}$ for the q-state Potts model on the complete graph with N vertices at inverse temperature $\beta \geq 0$, is the probability measure on $\{1, \ldots, q\}^N$ which to each $\sigma \in \{1, \ldots, q\}^N$ assigns probability

$$\mu^N_{q,\beta}(\sigma_1, \ldots, \sigma_n) = \frac{1}{Z^N_{q,\beta}} \exp\left(\frac{\beta}{2} \sum_{x=1}^{q} L_N(x)^2\right)$$

Here $Z^N_{q,\beta}$ is the normalizing constant.

THEOREM 22. (Ellis, Wang 1990) *Assume that $q \geq 3$, and define*

$$\beta_c(q) := \frac{2(q-1)}{q-2} \log(q-1).$$

Then we have the weak limit

$$\lim_{N\uparrow\infty} \mu_{q,\beta}\Big(L_N \in \cdot\Big)$$
$$= \begin{cases} \delta_{\frac{1}{q}(1,1,\dots,1)}, & \text{if } \beta < \beta_c(q) \\ \frac{1}{q}\sum_{\nu=1}^{q} \delta_{u(\beta,q)\, e_\nu + \frac{1-u(\beta,q)}{q}(1,1,\dots,1)}, & \text{if } \beta > \beta_c(q) \\ \lambda_0(q)\delta_{\frac{1}{q}(1,1,\dots,1)} + \frac{1-\lambda_0(q)}{q}\sum_{\nu=1}^{q} \delta_{u(\beta_c(q),q)\, e_\nu + \frac{1-u(\beta_c(q),q)}{q}(1,1,\dots,1)} & \text{if } \beta = \beta_c(q)\,, \end{cases}$$

where e_i is the unit vector in the i'th coordinate direction of $\mathbb{R}^q$.
The quantity $u(\beta, q)$ is well defined for $\beta \geq \beta_c(q)$. It is the largest solution of the mean field equation

$$u = \frac{1 - e^{-\beta u}}{1 + (q-1)e^{-\beta u}}$$

$u(\beta, q)$ *is strictly increasing in* β*, and we have* $u(q, \beta_c(q)) = \frac{q-2}{q-1}$*. The constant appearing at the critical point obeys the strict inequality* $0 < \lambda_0(q) < 1$*.*

$u(\beta, q)$ plays the role of an order parameter

The Ising model $q = 2$ can be recovered from the theorem by taking the formal limit $q \downarrow 2$.

Suppose that $\alpha(i) = \alpha(2)$ for $i = 2, \ldots, q$. Suppose that $\pi(i) = \pi(2)$ for $i = 2, \ldots, q$. Put $u = \pi(1) - \pi(2)$.

Then $\pi(1) = \bar{\pi}_u(1) := \frac{1+u(q-1)}{q}$, $\pi(i) := \bar{\pi}_u(i) = \frac{1-u}{q}$ for $i = 2, \ldots, q$ and we are left with

$$\frac{1-u}{1+u(q-1)} = \frac{\alpha(2)}{\alpha(1)} \exp(-\beta u)$$

or equivalently

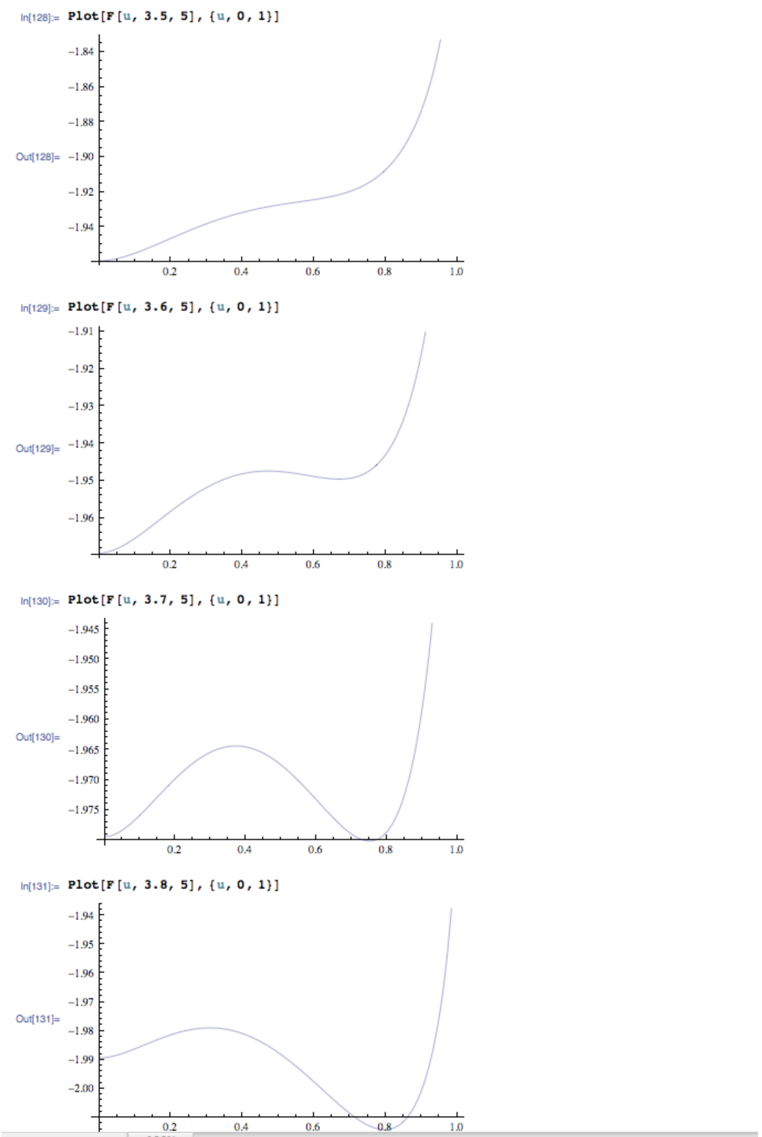
$$u = \frac{1 - \frac{\alpha(2)}{\alpha(1)} e^{-\beta u}}{1 + (q-1)\frac{\alpha(2)}{\alpha(1)} e^{-\beta u}} =: \psi_{\beta,q}(u)$$

This allows to treat the Potts model in the case where the particular value 1 is singled out, *under the assumption* that the minimizer satisfies $\pi(i) = \pi(2)$. This is not clear, even in the case of α being the equidistribution; it is proved in Ellis-Wang, and we will use it.

It is instructive to plot the function

$$F_{\beta,q}(u) := H_\beta(\bar{\pi}_u) + S(\bar{\pi}_u|\alpha_0)$$

for q fixed (e.g. $q = 5$) and different values of β increasing from 0.
The pictures below show $F_{\beta,q}(u)$ as a function of u for $q = 5$ when β is increased. We see first a *fold bifurcation* happening where a second minimum is produced, then we see the first order transition when the two minima are equal-depth. Increasing further β the positive minimum will become even deeper and stay the absolute minimizer.


In[128]:= Plot[F[u, 3.5, 5], {u, 0, 1}]
Out[128]=
In[129]:= Plot[F[u, 3.6, 5], {u, 0, 1}]
Out[129]=
In[130]:= Plot[F[u, 3.7, 5], {u, 0, 1}]
Out[130]=
In[131]:= Plot[F[u, 3.8, 5], {u, 0, 1}]
Out[131]=

Let us rewrite

$$\frac{\mu_{F,n}(\sigma(1)=\omega(1),\ldots,\sigma(k)=\omega(k))}{\mu_{F,n}(\sigma(1)=\omega'(1),\ldots,\sigma(k)=\omega'(k))}$$
$$=\frac{\prod_{i=1}^{k}\alpha(\omega_i)\sum_{\nu^{n-k}}\alpha^{n-k}(\Omega(\nu^{n-k}))e^{-nF(\frac{k}{n}L_k(\omega)+\frac{n-k}{n}\nu^{n-k})}}{\prod_{i=1}^{k}\alpha(\omega'_i)\sum_{\nu^{n-k}}\alpha^{n-k}(\Omega(\nu^{n-k}))e^{-nF(\frac{k}{n}L_k(\omega')+\frac{n-k}{n}\nu^{n-k})}}$$

This can be rewritten as

$$\frac{\prod_{i=1}^{k}\alpha(\omega_i)\dfrac{\sum_{\nu^{n-k}}\alpha^{n-k}(\Omega(\nu^{n-k}))\left[e^{-n\left[F(\frac{k}{n}L_k(\omega)+\frac{n-k}{n}\nu^{n-k})-F(\nu^{n-k})\right]}\right]e^{-nF(\nu^{n-k})}}{\sum_{\nu^{n-k}}\alpha^{n-k}(\Omega(\nu^{n-k}))e^{-nF(\nu^{n-k})}}}{\text{the same expression with } \omega' \text{ replacing } \omega}$$

Let us introduce the measure

$$\rho_{F,n,k}(\nu^{n-k}):=\frac{\alpha^{n-k}(\Omega(\nu^{n-k}))e^{-nF(\nu^{n-k})}}{\sum_{\tilde{\nu}^{n-k}}\alpha^{n-k}(\Omega(\tilde{\nu}^{n-k}))e^{-nF(\tilde{\nu}^{n-k})}}$$

Definition 23. *$F : \mathcal{P}(E) \mapsto \mathbb{R}$ is called differentiable if, for all $\alpha \in \mathcal{P}(E)$ there is a linear map $dF_\alpha : T(\mathcal{P}(E)) \mapsto \mathbb{R}$ on the tangent space $T(\mathcal{P}(E)) = \mathbb{R}\{\alpha' - \alpha | \alpha, \alpha' \in \mathcal{P}(E)\}$ such that*

$$F(\alpha') = F(\alpha) + dF_\alpha(\alpha' - \alpha) + \|\alpha' - \alpha\| r(\alpha', \alpha)$$

where $\alpha' \mapsto r(\alpha', \alpha)$ is continuous at $\alpha' = \alpha$ with $r(\alpha, \alpha) = 0$.

The tangent space

$$T(\mathcal{P}(E)) = \{(\pi(x)_{x \in E} | \sum_x \pi(x) = 0\}$$

is the space of signed measures with total mass 0.

Suppose that the Hamiltonian F is continuously differentiable.

Then, uniformly in α, α' we have

$$\sup_{\alpha,\alpha'} |F(\alpha + p(\alpha' - \alpha)) - F(\alpha) - p\, dF_\alpha(\alpha' - \alpha)| \leq Cpr(p)$$

where $r(p) \downarrow 0$ with $p \downarrow 0$. The uniformity in α, α' follows by the compactness of $\mathcal{P}(E)$.

Then

$$|F(\frac{k}{n}L_k(\omega) + \frac{n-k}{n}\nu^{n-k}) - F(\nu^{n-k}) - dF_{\nu^{n-k}}\frac{k}{n}\Big(L_k(\omega) - \nu^{n-k}\Big)| \leq C\frac{k}{n}r\Big(\frac{k}{n}\Big)$$

Hence

$$|nF(\frac{k}{n}L_k(\omega) + \frac{n-k}{n}\nu^{n-k}) - nF(\nu^{n-k}) - \sum_{i=1}^{k} dF_{\nu^{n-k}}(\delta_{\omega_i} - \nu^{n-k})| \leq Ckr\Big(\frac{k}{n}\Big)$$

This gives

$$\frac{\mu_{F,n}(\sigma(1)=\omega(1),\dots,\sigma(k)=\omega(k))}{\mu_{F,n}(\sigma(1)=\omega'(1),\dots,\sigma(k)=\omega'(k))}$$
$$\leq e^{2Ckr\left(\frac{k}{n}\right)}\frac{\prod_{i=1}^k \alpha(\omega_i)\sum_{\nu^{n-k}}\rho_{F,n,k}(\nu^{n-k})\prod_{i=1}^k e^{-dF_{\nu^{n-k}}(\delta_{\omega_i}-\nu^{n-k})}}{\text{the same with } \omega' \text{ replacing } \omega}$$

If $\mu \mapsto S(\mu|\alpha)+F(\mu)$ has a unique minimizer μ^* we have that, on continuous test-functions,

$$\rho_{F,n,k} \to \delta_{\mu_F^*}$$

(To take care of the difference between $n-k$ and n use a suitable normalization trick and put in what we know already!) This implies under the hypothesis that $\mu \mapsto S(\mu|\alpha)+F(\mu)$ has a unique minimizer μ^* we have

$$\frac{\mu_{F,n}(\sigma(1)=\omega(1),\dots,\sigma(k)=\omega(k))}{\mu_{F,n}(\sigma(1)=\omega'(1),\dots,\sigma(k)=\omega'(k))}$$
$$\to \frac{\prod_{i=1}^k \alpha(\omega_i)e^{-dF_{\mu_F^*}(\delta_{\omega_i}-\mu_F^*)}}{\text{the same with } \omega' \text{ replacing } \omega}$$

The measures appearing on the right hand side are formed with the following kernels.

Definition 24. *Assume that $F : \mathcal{P}(E) \mapsto \mathbb{R}$ is a real-valued function which is differentiable. Call the kernels*

$$\gamma_F(a|\mu) := \frac{e^{-dF_\mu(\delta_a-\mu)}\alpha(a)}{\sum_{b\in E} e^{-dF_\mu(\delta_b-\mu)}\alpha(b)}$$
$$= \frac{1}{1+\sum_{b\in E, b\neq a} e^{-dF_\mu(\delta_b-\delta_a)}\frac{\alpha(b)}{\alpha(a)}}$$

the associated mean-field kernel.

We have just proved the following Theorem.

THEOREM 25. *If $\mu \mapsto S(\mu|\alpha) + F(\mu)$ has a unique minimizer μ^* then, for fixed k, the marginals converge to the product measures given by the associated kernels, that is*

$$\lim_{n\uparrow\infty} \mu_{F,n}\Big(\sigma(1) = \omega(1), \ldots, \sigma(k) = \omega(k)\Big) = \prod_{i=1}^{k} \gamma_F(\omega(i)|\mu^*)$$

metric in the space of sequences in $\Omega = E^{\mathbb{N}} \equiv E^{\infty}$:

$$d(\sigma, \sigma') = \sum_{i=1}^{\infty} 2^{-i} 1_{\sigma(i) \neq \sigma'(i)}$$

metric on the space of probability measures $\mu, \mu' \in \mathcal{P}(E^{\infty})$ by

$$d(\mu, \mu') = \sum_{i=1}^{\infty} 2^{-i} \|\mu - \mu'\|_i$$

where

$$\begin{aligned} &\|\mu - \mu'\|_n \\ &:= \frac{1}{2} \sum_{(\omega(1),\ldots,\omega(n)) \in E^n} |\mu(\omega(1), \ldots, \omega(n)) - \mu'(\omega(1), \ldots, \omega(n))| \end{aligned}$$

With the metric d for measures on Ω we can reformulate the above statement as

$$\lim_{n\uparrow\infty} d\Big(\mu_{F,n}, \prod_{i=1}^{\infty} \gamma(\cdot|\mu^*)\Big) = 0$$

As we did for the distribution of the empirical distribution we want to formulate the analogue to the situation of non-unique minimizers.

THEOREM 26. *Denote by* $M^* \subset \mathcal{P}(E)$ *the set of minimizers of the map* $\nu \mapsto S(\nu|\alpha) + F(\nu)$. *Suppose that* M^* *is a finite set.*
Denote by

$$\mathcal{G}(F) := \{ \sum_{\nu \in M^*} p_\nu \prod_{i=1}^{\infty} \gamma(\cdot|\nu), p \in \mathcal{P}(M^*)\}$$

Then we have

$$\lim_{N \uparrow \infty} d(\mu_{F,N}, \mathcal{G}(F)) = 0$$

If there is a group T *acting as transformations* $\tau : \mathcal{P}(E) \mapsto \mathcal{P}(E)$ *for* $\tau \in T$ *which preserves the rate function,* $S(\nu|\alpha) + F(\nu) = S(\tau\nu|\alpha) + F(\tau\nu)$, *and* $M^* = \{\tau\nu^*, \tau \in T\}$ *then we have the symmetric expression*

$$\lim_{n \uparrow \infty} \mu_{F,n} = \frac{1}{|M^*|} \sum_{\nu \in M^*} \prod_{i=1}^{\infty} \gamma(\cdot|\nu)$$

On metastates, visibility and invisibility

(1) Metastates as empirical averages

(2) Metastates obtained by conditioning

(3) Examples

(4) Visibility vs. Invisibility in mean field models

with: Giulio Iacobelli

Lattice spin models with a quenched random Hamiltonian, examples

Edwards-Anderson spinglass

$$H = -\sum_{\langle i,j \rangle} J_{i,j} \sigma_i \sigma_j$$

Spins: $\sigma_i \in \{1, -1\}$

Random couplings: $J_{i,j} \sim \mathcal{N}(0,1)$, i.i.d.

Random field Ising model:

$$H = -\sum_{\langle i,j \rangle} \sigma_i \sigma_j - \varepsilon \sum_i \eta_i \sigma_i$$

Random fields: $\eta_i = \pm 1$ with equal probability, i.i.d.

The **metastate** is a concept to capture the asymptotic volume-dependence of the Gibbs states

$$"\mu(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z}"$$

Quenched (fixed) randomness $\eta = (\eta_i)_{i\in\mathbb{Z}^d}$.
Probability distribution $\mathbb{P}(d\eta)$
Infinite volume spin configuration $\sigma = (\sigma_i)_{i\in\mathbb{Z}^d}$
Infinite volume Hamiltonian $H^\eta(\sigma)$ (given in terms of an interaction Φ^η)

Fixing a boundary condition $\bar{\sigma}$, define the *finite-volume Gibbs states*

$$\mu_\Lambda^{\bar{\sigma}}[\eta](d\sigma)$$

in the finite volume $\Lambda \subset \mathbb{Z}^d$
restricting the terms of the Hamiltonian to $\Lambda = \Lambda_n = [-n, n]^d$

Common for *translation-invariant systems*:

to have convergence of the finite-volume states

$$\mu_{\Lambda_n}^{\bar{\sigma}}[\eta = 0](d\sigma) \to \mu^{\bar{\sigma}}(d\sigma)$$

as n gets large

Common for *disordered systems*:

not to have convergence of the finite-volume states:

$$\mu_{\Lambda_n}^{\bar{\sigma}}[\eta](d\sigma)$$

might have many cluster points when several Gibbs measures are available

Fix disorder η. Look at volume sequence

$$n \mapsto \mu_n[\eta]$$

Look at the empirical average (”histogramm”)

$$\kappa_N[\eta] := \frac{1}{N} \sum_{n=1}^{N} \delta_{\mu_n[\eta]}$$

Def. Suppose the following limit exists in the sense of local topologies

$$\kappa[\eta] := \lim_{N \uparrow \infty} \kappa_N[\eta]$$

for a.e. realization η. Then it is is called an

Newman-Stein metastate or empirical metastate.

Problem: It might depend on subsequence.

Look at the probability distribution of

$$(\mu_n[\eta], \eta)$$

under the governing measure $\mathbb{P}$ of the disorder variable

Def. Suppose that a limit exists for this random pair in the sense of weak convergence.
Call the resulting limiting distribution $K(d\mu, d\eta)$

Then the conditional measure

$$\kappa^{\text{AW}}[\eta](d\mu) := K(d\mu|\eta)$$

is called **Aizenman-Wehr or conditional metastate**.

Existence of the limit K is guaranteed (only) for subsequences of n's
independence of the limit of subsequence is not proved in general

Theorem. (Newman Stein 97)

Take sufficiently sparse sequences n_k, $k = 1, 2, \ldots$ of the subsequence of n's in the construction of the Aizenman-Wehr metastate $\kappa^{\mathsf{AW}}[\eta]$.

Then, for sufficiently sparse subsequences N_l, the corresponding Newman-Stein metastate converges to the Aizenman-Wehr metastate:

$$\lim_{l\uparrow\infty} \frac{1}{N_l} \sum_{k=1}^{N_l} \delta_{\mu_{n_k}[\eta]} = \kappa^{\mathsf{AW}}[\eta]$$

Bovier book: Statistical mechanics of disordered systems

Bovier, Gayrard: Hopfield with many patterns and infinitely many Gibbs measures

van Enter, Bovier, Niederhauser: Hopfield model with Gaussian fields (continuous symmetry)

van Enter, Netocny, Schaap: Ising ferromagnet on lattice with random boundary conditions

Arguin, Damron, Newman, Stein (2009): ”Metastate-version” of uniqueness of groundstate for lattice-spinglass in 2 dimensions

Theorem. Arguin, Damron, Newman, Stein (2009)

$$H = -\sum_{\langle i,j\rangle} J_{i,j}\sigma_i\sigma_j$$

$$\sigma_i \pm 1$$

$$J_{i,j} \sim \mathcal{N}(0, \sigma^2)$$

Denote $\alpha_J^{(L)} = \pm\sigma_J^{(L)}$ the ground state pair in a box of height L, width $2L$ with base on the x-axis, with free boundary conditions in vertical directions, periodic in horizontal directions.

Take any two distributional subsequence limits of the random pair $(J, \alpha_J^{(L)})$,

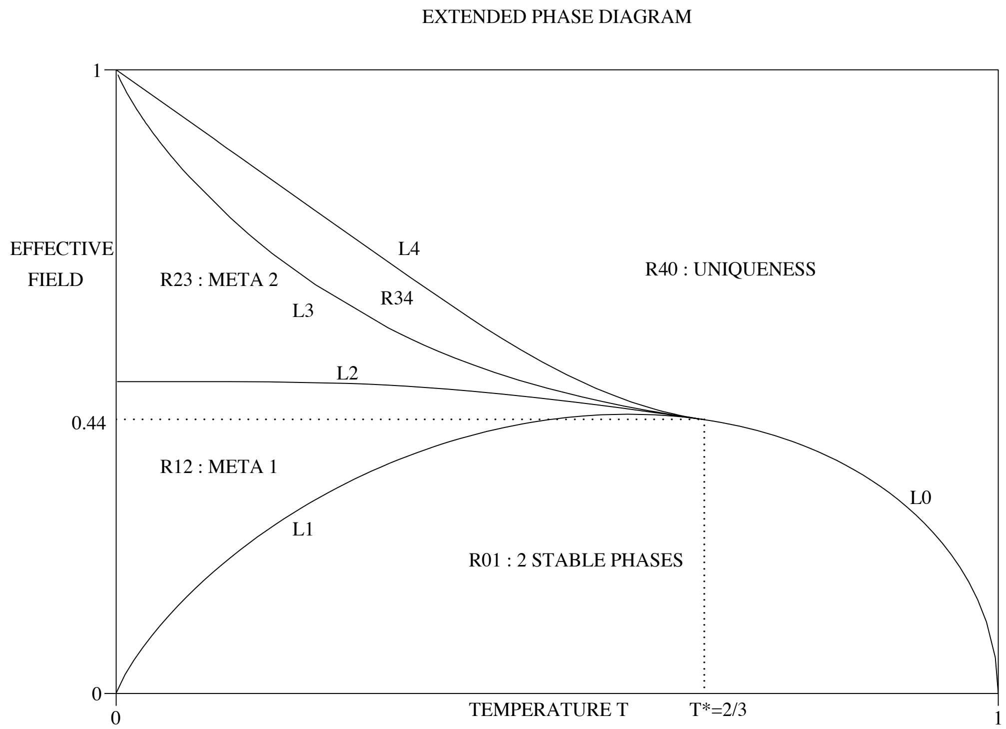
denoted by $\kappa^*(dJ, d\alpha)$, $\bar{\kappa}^*(dJ, d\alpha)$.

Then for, $\mathbb{P}$- a.e. J with $\kappa^*(d\alpha|J) \times \bar{\kappa}^*(d\alpha'|J)$-probability one has $\alpha = \alpha'$.

(Strengthening of Newman, Stein CMP 2001)

Mean-Field Random Field Ising Model

$$\mu_n[\eta]\left((\sigma_i)_{i=1,\dots,n}\right) = \frac{1}{Z_n[\eta]} \exp\left(\frac{\beta}{2n}\sum_{1\le i,j\le n}\sigma_i\sigma_j + \beta\varepsilon\sum_{1\le i\le n}\eta_i\sigma_i\right)$$



Theorem. (Külske 97)

$$\lim_{N\uparrow\infty}\frac{1}{N}\sum_{n=1}^{N}F(\mu_n[\eta]) =^{\text{law}} n_\infty F\left(\mu_\infty^+[\eta]\right)+(1-n_\infty)F\left(\mu_\infty^-[\eta]\right)$$

where n_∞ is a 'fresh' random variable, independent of η on the r.h.s., with arcsine-distribution (that is $\mathbb{P}\left[n_\infty < x\right] = \frac{2}{\pi}\arcsin\sqrt{x}$).

If n_k are chosen sufficiently sparse, the Newman-Stein metastate is given by $\frac{1}{2}F\left(\mu_\infty^+[\eta]\right)+\frac{1}{2}F\left(\mu_\infty^-[\eta]\right)$, for almost every realization of η.

Floris Takens:

Dynamical systems examples of low dimensional systems with "historic behavior", i.e.

$\frac{1}{T}\int_0^T \delta_{\gamma(t)} dt$ does not converge for a flow $\gamma(t) \in \mathbb{R}^2$

oscillating between two attractors, slower and slower

Spin variables: $\sigma(i)$ taking values in a finite set E
Disorder variable: $\eta(i)$ taking values in a finite set E'
Sites: $i \in \{1, 2, \ldots, n\}$

$$\mathcal{P}(E) = \{\text{set of probability measures on } E\}$$

$$= \{(p(a))_{a \in E} : p(a) \geq 0, \sum_{a \in E} p(a) = 1\}$$

$$L_n = \text{empirical distribution} = \frac{1}{n} \sum_{i=1}^{n} \delta_{\sigma(i)} \in \mathcal{P}(E)$$

$$F : \mathcal{P}(E) \to \mathbb{R},$$

twice continuously differentiable.

Local a priori measures $\alpha[b] \in \mathcal{P}(E)$
for any possible type of the disorder $b \in E'$.

Mean-field interaction F

A priori measures $\alpha = (\alpha[b])_{b \in E'}$

Disorder distribution $\pi \in \mathcal{P}(E')$

Definition 27. *The **disorder-dependent finite-volume Gibbs measures** are*

$$\mu_{F,n}[\eta(1), \ldots, \eta(n)](\sigma(1) = \omega(1), \ldots, \sigma(n) = \omega(n))$$
$$= \frac{1}{Z_{F,n}[\eta(1), \ldots, \eta(n)]} \exp\left(-nF\left(L_n^{\omega}\right)\right) \prod_{i=1}^{n} \alpha[\eta_i](\omega_i)$$

Frozen disorder: $\eta(i) \sim \pi$ i.i.d. over sites i

***Definition* 28.** *Assume that, for every bounded continuous* $G : \mathcal{P}(E^{\infty}) \times (E')^{\infty} \to \mathbb{R}$ *the limit*

$$\lim_{n \uparrow \infty} \int \mathbb{P}(d\eta) G(\mu_n[\eta], \eta) = \int J(d\mu, d\eta) G(\mu, \eta)$$

exists. Then the conditional distribution $\kappa[\eta](d\mu) := J(d\mu|\eta)$ *is called the* AW-metastate on the level of the states*.*

Volume of b-like sites, given η:

$$\Lambda_n(b) = \{i \in \{1,2,\ldots,n\}; \eta(i) = b\}$$

Frequency of the b-like sites:

$$\widehat{\pi}_n(b) = \frac{|\Lambda_n(b)|}{n}$$

empirical spin-distribution on the b-like sites:

$$\widehat{L}_n(b) = \frac{1}{|\Lambda_n(b)|} \sum_{i \in \Lambda_n(b)} \delta_{\sigma(i)}$$

vector of empirical distributions:

$$\widehat{L}_n = (\widehat{L}_n(b))_{b \in E'}$$

total empirical spin-distribution

$$L_n = \sum_{b \in E'} \widehat{\pi}_n(b) \widehat{L}_n(b)$$

***Definition* 29.** *Consider the free energy minimization problem*

$$\widehat{\nu} \mapsto \Phi[\pi](\widehat{\nu})$$

on $\mathcal{P}(E)^{E'}$, with the free energy functional

$$\Phi : \mathcal{P}(E') \times \mathcal{P}(E)^{|E'|} \to \mathbb{R}$$

$$\Phi[\widehat{\pi}](\widehat{\nu}) = F\left(\sum_{b\in E'} \widehat{\pi}(b)\widehat{\nu}(b)\right) + \sum_{b} \widehat{\pi}(b) S(\widehat{\nu}(b)|\alpha[b])$$

where $S(p_1|p_2) = \sum_{a\in E} p_1(a)\log\frac{p_1(a)}{p_2(a)}$ is the relative entropy.

Non-degeneracy condition 1:

$\widehat{\nu} \mapsto \Phi[\pi](\widehat{\nu})$ has a finite set of minimizers $M^* = M^*(F,\alpha,\pi)$ with positive curvature.

Let $\widehat{\nu}_j$ be a fixed element in M^*. Let us consider the linearization of the free energy functional at the fixed minimizers as a function of $\tilde{\pi}$ around π, which reads

$$\Phi[\tilde{\pi}](\widehat{\nu}_j) - \Phi[\pi](\widehat{\nu}_j) = -B_j[\tilde{\pi} - \pi] + o(\|\tilde{\pi} - \pi\|)$$

where

$$B_j[\tilde{\pi} - \pi] = -\left(dF_{\pi \cdot \widehat{\nu}_j}\left(\sum_b (\tilde{\pi}(b) - \pi(b))\widehat{\nu}_j(b)\right) + \sum_b [\tilde{\pi}(b) - \pi(b)] S(\widehat{\nu}_j(b)|\alpha[b])\right)$$

This defines an affine function on the tangent space of field type measures $T\mathcal{P}(E')$ (i.e. vectors which sum up to zero, isomorphic to $\mathbb{R}^{|E'|-1}$), for any j.

Non-degeneracy condition 2:

No different minimizers j, j' have the same $B_j = B_{j'}$

***Definition* 30.** *Call B_j the **stability vector of** $\widehat{\nu}_j$ and call*

$$R_j := \{x \in T\mathcal{P}(E'), \langle x, B_j \rangle > \max_{k \neq j} \langle x, B_k \rangle\}$$

***stability region of** $\widehat{\nu}_j$.*

Lemma 1. Condition 2 implies that Lebesgue$\left(\left(\bigcup_{j=1,\dots,k} R_j\right)^c\right) = 0$

Lemma 2. $R_j \neq \emptyset \Leftrightarrow B_j \in \mathrm{ex}(\mathcal{H}_{\mathrm{conv}}\{B_1, \dots, B_k\})$.

Proof of Lemma 1. If $j \mapsto \langle x, B_j \rangle$ has no unique maximizer for fixed x, then $\exists j \neq k$ such that $\langle x, B_j - B_k \rangle = 0$.

For fixed j, k this set of x's is a hyperplane (hence a measure zero set) since $B_j \neq B_k$.

Proof of Lemma 2.

$'' \Rightarrow ''$ by contradiction.

If B_j is not extremal, then

$$B_j = \sum_{i:i\neq j} \alpha_i B_i, \qquad \sum_{i=1}^{k} \alpha_i = 1$$

$$x \in R_j \Rightarrow \langle x, B_j \rangle = \sum_{i} \alpha_i \langle x, B_j \rangle > \sum_{i=1}^{k} \alpha_i \langle x, B_i \rangle = \langle x, B_j \rangle$$

Proof of Lemma 2.

$'' \Leftarrow ''$

Given $B_j \notin \mathcal{H}_{\text{conv}}\{B_1, \dots, B_{j-1}, B_{j+1}, \dots, B_k\}$ chose coordinates

$$B_j = (0, \dots, 0, B_{j,d})$$
$$B_i = (B'_i, B_{i,d})$$

with $B_{j,d} > 0$ and $B_{i,d} \leq 0$ for $i \neq j$.

With $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$

$$R_j = \{x \in \mathbb{R}^d : \forall i \neq j \text{ holds } \underbrace{\langle x, B_j - B_i \rangle}_{\langle x', B'_j - B'_i \rangle + x_d (B_{j,d} - B_{i,d})} > 0\}$$
$$= \{x \in \mathbb{R}^d : x_d > \max_{i : i \neq j} \frac{\langle x', B'_i - B'_j \rangle}{B_{j,d} - B_{i,d}}\} \neq \emptyset$$

$\square$

***THEOREM* 31.** *(Iacobelli, Külske, JSP 2010) Assume that the model satisfies the non-degeneracy assumptions 1 and 2. Define the weights*

$$w_j := \mathbb{P}_\pi(G \in R_j)$$

where G taking values in $T\mathcal{P}(E')$ is a centered Gaussian variable with covariance

$$C_\pi(b,b') = \pi(b)1_{b=b'} - \pi(b)\pi(b')$$

Then the Aizenman-Wehr metastate on the level of the states equals

$$\kappa[\eta](d\mu) = \sum_{j=1}^{k} w_j \delta_{\mu_j[\eta]}(d\mu)$$

where $\mu_j[\eta] := \prod_{i=1}^{\infty} \gamma[\eta(i)](\,\cdot\,|\pi\widehat{\nu}_j)$ with

$$\gamma[b](a|\nu) = \frac{e^{-dF_\nu(a)}\alpha[b](a)}{\sum_{\bar{a}\in E} e^{-dF_\nu(\bar{a})}\alpha[b](\bar{a})}$$

Def. Call

$$M^{**} = \{\widehat{\nu} \in M^* : w_{\widehat{\nu}} > 0\}$$

the **visible pure phases** in the pure phases M^*.

Comment. With the bijection

$$\begin{aligned} B_{\cdot} : M^* &\to T\mathcal{P}(E') \\ \widehat{\nu} &\mapsto B_{\widehat{\nu}} \end{aligned}$$

we have

$$M^{**} = (B_{\cdot})^{-1}\Big(\mathsf{ex}(\mathcal{H}_{\mathsf{conv}}(B_{\cdot}(M^*))\Big)$$

Corollary 32. *Suppose that the system admits precisely two pure phases, i.e. $|M^*| = 2$, and $|E|, |E'|$ are finite but arbitrary.*

Then the metastate is the symmetric mixture between the two, i.e.

$$\kappa[\eta](d\mu) = \frac{1}{2}\delta_{\mu_1[\eta]}(d\mu) + \frac{1}{2}\delta_{\mu_2}[\eta](d\mu)$$

Proof: $R_1 = -R_2$

***Corollary* 33.** *Suppose that the random field is two-valued, i.e.* $|E'| = 2$*, and the number of pure phases* $|M^*| \geq 2$ *arbitrary. Then the set of visible states has two elements and* $w(\widehat{\nu}) = \frac{1}{2}$ *for both elements* $\widehat{\nu} \in M^{**}$*.*

Proof: a convex set in one dimension has two extremal points.

Definition 34. *Assume that, for every bounded continuous* $F : \mathcal{P}(\mathcal{P}(E)) \times (E')^\infty$ *the limit*

$$\lim_{n\uparrow\infty} \int \mathbb{P}(d\eta) F(L_n(\mu_n[\eta]), \eta) = \int K(d\rho, d\eta) F(\rho, \eta)$$

exists. Then the conditional distribution $\bar{\kappa}[\eta](d\rho) := K(d\rho|\eta)$ *is called the* metastate on the level of the empirical spin-distribution*.*

THEOREM 35. *Under the two non-degeneracy conditions, we have*

$$\bar{\kappa}[\eta](d\rho) = \sum_{j=1}^{k} w_j \delta_{\delta_{\pi\widehat{\nu}_j}}(d\rho)$$

for $\mathbb{P}_\pi$*-a.e.* η*.*

The minimizers of the variational problem above must satisfy the consistency (mean-field) equations

$$\widehat{\nu}[b](a) = \gamma[b](a|\pi \cdot \widehat{\nu})$$

which are coupled over $b \in E'$. Summing over these indices one gets the mean-field equation for the total empirical mean $\nu = \pi \cdot \widehat{\nu}$ of the form

$$\nu(a) = \sum_{b \in E'} \pi(b)\gamma[b](a|\nu)$$

Lemma 36. *Define the function* $\widehat{\Gamma} : \mathcal{P}(E) \to \mathcal{P}(E)^{E'}$ *by the r.h.s. of the mean field equation, namely*

$$\widehat{\Gamma}(\nu) = \Big(\gamma[b](\cdot|\nu)\Big)_{b \in E'}$$

Define the function $\widehat{B} : \mathcal{P}(E) \to T\mathcal{P}(E')$ *by*

$$\widehat{B}_\nu[b] = \log \sum_{a \in E} e^{-dF_\nu(a)} \alpha[b](a) - C$$

$$C = \frac{1}{|E'|} \sum_{b \in E'} \log \sum_{a \in E} e^{-dF_\nu(a)} \alpha[b](a)$$

Then, for all $\widehat{\nu} \in M^*$ *we have that*

$$\widehat{\nu} = \widehat{\Gamma}(\pi\widehat{\nu})$$

$$B_{\widehat{\nu}} = \widehat{B}_{\pi\widehat{\nu}}$$

π does not enter (but through the question which minimizer $\widehat{\nu}$ appears.)

Quadratic Curie Weiss Random Field Ising model with

$$F(\nu) = -\beta(\nu(+)^2 + \nu(-)^2)$$

Any possible local single-site measure α can be described as

$$\alpha[h](\sigma_i) = \frac{e^{h\sigma_i}}{2\cosh h}$$

Any $\nu = \nu_m$ can be described in terms of its mean value $\nu_m(+) - \nu_m(-) = m$.
Fix $E' = \text{supp}(\pi) = \{\alpha_h : h \in \{h_1, h_2, \ldots, h_L\}\}$

$$\widehat{B}_{\nu_m} = \begin{pmatrix} \log \frac{\cosh(\beta m + h_1)}{\cosh h_1} \\ \ldots \\ \log \frac{\cosh(\beta m + h_L)}{\cosh h_L} \end{pmatrix} - \frac{1}{L}\sum_{j=1}^{L} \log \frac{\cosh(\beta m + h_j)}{\cosh h_j} \begin{pmatrix} 1 \\ \ldots \\ 1 \end{pmatrix}$$

***Lemma* 37.** *Let $E' \subset \mathbb{R}$, $2 \leq |E'| < \infty$. Then the map $m \mapsto \widehat{B}_{\nu_m}$ is injective.*

Random-field Ising model to a non-quadratic Hamiltonian

$$F(\nu) = G(\nu(+) - \nu(-))$$

and general local measures $\alpha = (\alpha[h])_{h \in E'}$

mean field equation:

$$m = \sum_{i=1}^{L} \pi(h_i) \tanh(-G'(m) + h_i)$$

The stability vector becomes

$$\widehat{B}_{\nu_m}[h_i] := \log \frac{\cosh(-G'(m) + h_i)}{\cosh h_i} - \frac{1}{L} \sum_{j=1}^{L} \log \frac{\cosh(-G'(m) + h_j)}{\cosh h_j}$$

$m \mapsto \widehat{B}_{\nu_m}$ is injective if $m \mapsto G'(m)$ is injective

One can create interactions G with two minima without symmetry, by looking at the equal-depth condition for the free energy.

There both minima would get the same weight in the metastate necessarily.

precisely two minimizers not related by symmetry were proved to occur (even) for the (symmetric) model

$$G(m) = -\frac{\beta m^2}{2}$$

$E = E' = \{1, -1\}$, asymmetric random field distribution
$\pi(1) = \frac{1+\alpha}{2} = 1 - \pi(-1)$
for the region R_{34} in phase space (Külske-LeNy CMP 2007) for a suitable choice of $\alpha = \alpha(\beta, \varepsilon) > 0$.

Let us take the Potts model with quadratic interaction

$$F(\nu) = -\frac{\beta}{2}(\nu(1)^2 + \cdots + \nu(q)^2)$$

in the presence of the local single-site measures $\alpha[b](\sigma_i)$ (specified below) where we write

$$E' = \text{supp}(\pi) = \{\alpha[b] : b \in \{b_1, b_2, \ldots, b_L\}\}$$

Then we have for the stability vector

$$\hat{B}_\nu = \begin{pmatrix} \log \sum_{a=1}^q e^{\beta\nu(a)}\alpha[b_1](a) \\ \ldots \\ \log \sum_{a=1}^q e^{\beta\nu(a)}\alpha[b_L](a) \end{pmatrix} - \frac{1}{L}\sum_{j=1}^{L} \log \sum_{a=1}^{q} e^{\beta\nu(a)}\alpha[b_j](a) \begin{pmatrix} 1 \\ \ldots \\ 1 \end{pmatrix}$$

Let us take $E \equiv E'$ and π to be the equidistribution and switch to the specific case $\alpha[b](a) = \frac{e^{B1_{b=a}}}{e^B+q-1}$ (random field with homogenous intensity). The kernels become

$$\gamma[b](a|\nu) = \frac{e^{\beta\nu(a)+B1_{a=b}}}{\sum_{\bar{a}\in E} e^{\beta\nu(\bar{a})+B1_{\bar{a}=b}}}$$

We will be looking at measures in $\nu_{j,u} \in \mathcal{P}(E)$ of the form $\nu_{j,u}(j) = \frac{1+u(q-1)}{q}$, $\nu_{j,u}(i) = \frac{1-u}{q}$ for $i \neq j$. The stability vector for $\nu_{1,u}$ is given by

$$\widehat{B}_{\nu_{1,u}} = \begin{pmatrix} \frac{q-1}{q} \log \frac{e^{\beta u + B} + q - 1}{e^{\beta u} + e^{B} + q - 2} \\ -\frac{1}{q} \log \frac{e^{\beta u + B} + q - 1}{e^{\beta u} + e^{B} + q - 2} \\ \dots \\ -\frac{1}{q} \log \frac{e^{\beta u + B} + q - 1}{e^{\beta u} + e^{B} + q - 2} \end{pmatrix}$$

the other ones are related by symmetry. We note that the first entry is strictly positive while the other entries are negative (for $B > 0$ and $u > 0$).

mean-field equation for u:

$$u = \frac{e^{\beta u}}{e^{\beta u} + e^{B} + (q-2)} - \frac{1}{e^{\beta u + B} + (q-1)}$$

$u = 0$ is always a solution

for $B = 0$: mean-field equation for Potts without disorder

the non-trivial solution u is to be chosen iff $\Phi[\pi](\widehat{\Gamma}(\nu_{j,u})) < \Phi[\pi](\widehat{\Gamma}(\nu_{j,u=0}))$

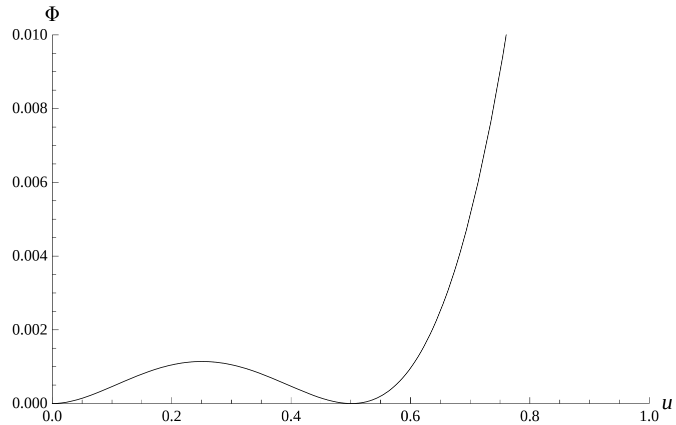
$$\begin{aligned}
&\Phi[\pi](\widehat{\Gamma}(\nu_{j,u})) - \Phi[\pi](\widehat{\Gamma}(\nu_{j,u=0}) \\
&= \log \frac{e^{B} + q - 1}{e^{\beta u} + e^{B} + q - 2} + \frac{\beta(q-1)}{2q} u^2 + \frac{\beta}{q} u - \frac{1}{q} \log \frac{e^{\beta u + B} + q - 1}{e^{\beta u} + e^{B} + q - 2}
\end{aligned}$$

$B = 0$: first order transition at the critical inverse temperature $\beta = 4 \log 2$

B takes small enough positive values: line in the space of temperature and coupling strength B of an equal-depth minimum at $u = 0$ and a positive value of $u = u^*(\beta, q)$

Along this line the set of Gibbs measures is strictly bigger then the set of states which are seen under the metastate.

The Plot shows the graph of $u \mapsto \Phi[\pi](\hat{\Gamma}(\nu_{j,u}))$ for $B = 0.3, q = 3, \beta = 4 \log 2 + 0.03203$ at which there is the first order transition.



$$\kappa[\eta](d\mu) = \frac{1}{3}\sum_{j=1}^{3} \delta_{\mu_j[\eta]}$$

with

$$\mu_j[\eta] = \prod_{i=1}^{\infty} \gamma[\eta(i)](\,\cdot\,|\nu_{j,u=u^*(\beta,q)})$$

since $\widehat{B}_{\nu_{1,u=0}} = 0$ lies in the convex hull of the three others

Concentration of the total empirical spin vector follows from finite-volume Sanov:

$$\mu_{F,n}[\eta(1),\dots,\eta(n)](d(L_n,\pi M^*)\geq\varepsilon)$$

$$\leq \prod_{b\in E'}(n\widehat{\pi}_n(b)+1)^{2|E|}\exp\left(-n\inf_{\substack{\widehat{\nu}\in\widehat{M}_n:\\ d(\widehat{\pi}_n\widehat{\nu},\pi M^*)\geq\varepsilon}}\Phi[\widehat{\pi}_n](\widehat{\nu})+n\inf_{\widehat{\nu}'\in\widehat{M}_n}\Phi[\widehat{\pi}_n](\widehat{\nu}')\right)$$

$\widehat{\pi}_n$: empirical field-type distribution

This explains the importance of the spin-rate-function $\Phi[\eta](\widehat{\nu})$

for not too atypical $\widehat{\pi}_n$.

How to get weights w_j?

Fluctuations of type-empirical distribution on CLT-scale:

$$X_{[1,n]}[\eta] = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\delta_{\eta_i} - \pi) \to G$$

Define n-dependent *good-sets* $\mathcal{H}_n^{\delta_n}$ of the realization of the randomness

$$\mathcal{H}_{i,n}^{\delta_n} := \left\{ \eta \in (E')^n : X_{[1,n]}[\eta] \in R_{i,\delta_n} \right\}$$

$$\mathcal{H}_n^{\delta_n} := \bigcup_{i=1}^{k} \mathcal{H}_{i,n}^{\delta_n}$$

where $R_{i,\delta_n} := \{x \in T\mathcal{P}(E') : \langle x, B_i \rangle - \max_{k \neq i} \langle x, B_k \rangle > \delta_n\}$, and

(a) $\delta_n \downarrow 0$, but

(b) $\sqrt{n}\, \delta_n \uparrow \infty$

(a) Get full proba of $\mathcal{H}_n^{\delta_n}$ in the limit of $n \uparrow \infty$.

(b) Have concentration of $\widehat{L}_n$ around a given minimizer $\widehat{\nu}_j$ on $\mathcal{H}_{j,n}^{\delta_n}$.

Suppose F is a local function, depending on m coordinates of spins and random fields.

Then:

$$\lim_{n\uparrow\infty}\int_{\mathcal{H}_{j,n}^{\delta_n}}\mathbb{P}_\pi(d\eta)F(\mu_n[\eta],\eta)=w_j\int_{(E')^m}\pi^{\otimes m}(d\eta)F\Big(\prod_{i=1}^m\gamma[\eta(i)](\,\cdot\,|\pi\widehat{\nu}_j),\eta\Big)$$

Productification with only local influence of randomness conditional on stability region R_j.

Spin variables: $\sigma = (\sigma_x)_{x \in \mathbb{Z}^d} \in E^{\mathbb{Z}^d}$

$E = \{-1, 1\}$, finite set, sphere, $\mathbb{R}$

Interaction potential: $\Phi = (\Phi_A(\sigma_A))_{A \subset \mathbb{Z}^d}$

Hamiltonian $\equiv$ Formal energy function: $H(\sigma) = \sum_{A : A \subset \mathbb{Z}^d} \Phi_A(\sigma)$

Example: $H(\sigma) = -\beta \sum_{<x,y>} \sigma_x \sigma_y - h \sum_x \sigma_x$

Finite volume Gibbs measures in volumes $\Lambda \subset \mathbb{Z}^d \equiv$ specification:

$$\gamma_\Lambda(\sigma_\Lambda | \sigma_{\Lambda^c}) := \frac{\exp\left(-\sum_{A : A \cap \Lambda \neq \emptyset} \Phi_A(\sigma_\Lambda \sigma_{\Lambda^c})\right) \prod_{i \in \Lambda} \alpha(\sigma_i)}{\sum_{\tilde{\sigma}_\Lambda} \exp\left(-\sum_{A : A \cap \Lambda \neq \emptyset} \Phi_A(\tilde{\sigma}_\Lambda \sigma_{\Lambda^c})\right) \prod_{i \in \Lambda} \alpha(\tilde{\sigma}_i)}$$

Tasks of Equilibrium Statistical Mechanics :

• Usually: Given Φ, characterize **Infinite volume Gibbs measures** μ:

(DLR-equation) $\quad \mu(\sigma_\Lambda | \sigma_{\Lambda^c}) = \gamma_\Lambda(\sigma_\Lambda | \sigma_{\Lambda^c})$

• Sometimes: **Given** μ **find** Φ !

The set of Gibbs measures $\mathcal{G}(\Phi)$ can consist of more than one point.

This we call **phase transition**

The extremal points of $\mathcal{G}(\Phi)$ are called **pure phases**

Recall Formentin talk and note without proof:

Non-reconstructability of free boundary condition Potts measure
is equivalent to extremality of this measure in the Gibbs measures
(no information transport from infinity on the tree)

Definition* 38.** *1.* ***Dobrushin's interdependence matrix *for the specification γ^Φ is the matrix whose entries are given by*

$$C_{ij}(\gamma^\Phi) = \sup_{\zeta,\eta\in\Omega;\, \zeta_{j^c}=\eta_{j^c}} \|\gamma_i(\cdot|\zeta) - \gamma_i(\cdot|\eta)\|.$$

2. The specification γ^Φ satisfies ***Dobrushin's condition*** *if it is quasilocal and the Dobrushin constant satisfies*

$$c(\gamma^\Phi) = \sup_{i\in G} \sum_{j\in G} C_{ij}(\gamma^\Phi) < 1.$$

Dobrushin's condition implies uniqueness of the Gibbs measure corresponding to γ^Φ, see Georgii

***Proposition* 39.** *Let*

$$\sup_{i\in G} \sum_{A\in i} (|A|-1)\delta(\Phi_A) < 2,$$

where $\delta(\Phi_A) := \sup_\sigma \Phi_A(\sigma) - \inf_w \Phi_A(w)$. Then γ^Φ satisfies Dobrushin's condition.

Classic notion: A lattice measure μ is **Gibbs** iff exists a potential Φ s.t.

(i) Φ reproduces correct conditional probabilities (DLR)

(ii) Φ is **absolutely summable**, i.e.

$$\sum_{A \ni x} \sup_{\sigma_A} |\Phi_A(\sigma_A)| < \infty$$

Notion of **Gibbs** measure is related to **continuity of conditional probabilities** (w.r.t. product topology):

ξ a **good configuration** for μ

$$:\Leftrightarrow \quad \sup_{\substack{\sigma^+,\sigma^- \\ \Lambda:\Lambda\supset V}} \left| \mu(\tilde{\sigma}_x \middle| \xi_{\mathbf{V}\setminus\mathbf{x}} \sigma^+_{\Lambda\setminus V}) - \mu(\tilde{\sigma}_x \middle| \xi_{\mathbf{V}\setminus\mathbf{x}} \sigma^-_{\Lambda\setminus V}) \right| \to 0$$

with $V \uparrow \mathbb{Z}^d$, for any site $x \in \mathbb{Z}^d$, for any $\tilde{\sigma}_x$

Known: μ Gibbs iff every configuration is good (Kozlov)

Georgii-book: Gibbs measures and phase-transitions (1988)
van Enter, Fernandez, Sokal: Regularity properties and Pathologies of position-space renormalization group transformations (JSP 1993)

Decimation transformation on the lattice

Take μ_β^+ a low-temperature state on the lattice for Ising

β large, $h = 0$

$\mu'_\beta =$ restriction of μ_β to a sublattice

Known: μ'_β is not a Gibbs measure (renormalization group pathology)

a potential with weaker summability properties (only μ'_β-a.s.) exists

Proof: hard work: multiscale cluster expansions (renormalization group)

Bricmont, Kupiainen, Lefevere (98)

Maes, Redig, Shlosman, van Moffaert (2000)

Bertini, Cirillo, Olivieri (2006)

Initial system: Nearest neighbor Ising model $\mu_{t=0} := \mu^{+}_{\beta,h}$

The dynamics:

symmetric independent spin-flips:

$$\mu_t(\eta_\Lambda) = \int \mu_{t=0}(d\sigma_\Lambda) \prod_{x\in\Lambda} p_t(\sigma_x, \eta_x)$$

transition kernel for rate-1 flips: $p_t(+,+) = \frac{1}{2}(1 + e^{-2t})$

$\Big(p_t(+,+) = p_t(-,-) = 1 - p_t(+,-) = 1 - p_t(-,+)\Big)$

$\Rightarrow$ trivial infinite-time limiting measure (locally):

$$\lim_{t\uparrow\infty} \mu_t = \bigotimes_{x\in\mathbb{Z}^d} \frac{1}{2}\Big(\delta_+ + \delta_-\Big)$$

$\mu_{\beta,h=0,t}$ **fails to be Gibbs for β large, t large** due to ”hidden phase transitions”

***THEOREM* 40.**

Assume $\beta << \beta_d$ (high-temperature regime). Then:

(0) $\mu_{\beta,h;t}$ *is a Gibbs measure for all* $t > 0$

Assume $\beta >> \beta_d$ (low-temperature regime). Then:

(i) $\mu_{\beta,h;t}$ *is a Gibbs measure for all* $0 \leq t \leq t_0(\beta, h)$

(ii) *If* $h > 0$*, then* $\mu_{\beta,h;t}$ *is a Gibbs measure for all* $t \geq t_1(h)$

(iii) *If* $h = 0$*, then* $\mu_{\beta,h;t}$ *is* ***not*** *a Gibbs measure for all* $t \geq t_2(\beta)$

(iv) *For* $d \geq 3$*, if* $0 < h \leq h(\beta)$*, then*

$\mu_{\beta,h;t}$ *is* ***not*** *a Gibbs measure for all* $t_3(\beta, h) \leq t \leq t_4(\beta, h)$

vE-F-dH-R prove moreover:

Results hold more generally for non-independent high-temperature dynamics (uses expansions)

- van Enter, Fernandez, den Hollander, Redig (CMP 2002): Ising-spinflip
- Külske, Redig (PTRF 2006):
  Unbounded Continuous variables under Diffusions
- Külske, LeNy (CMP 2007):
  Mean-Field Ising - symmetry breaking in bad configurations
- Külske, Opoku (EJP, JMP 2008) Goodness of Gibbsianness,
  Lattice vs. Meanfield
- van Enter, Ruszel (JMP 2008, SPA 2009):
  Bounded Continuous variables (circle) under Diffusions
- Enter, Külske, Opoku, Ruszel (BJPS 2010):
  Gibbs-non-Gibbs properties for n-vector lattice and mean-field models
- van Enter, Fernandez, den Hollander, Redig (arXiv 2010):
  A large-deviation view on dynamical Gibbs-non Gibbs transitions
- Ermolaev, Külske (arXiv 2010):
  Low temp. dynamics of Mean-Field Ising - periodic orbits, multiple histories

$$p_t(\sigma_x, \eta_x) = \frac{e^{h_t \eta_x \sigma_x}}{2\cosh h_t}, \quad h_t = \frac{1}{2}\log\frac{1+e^{-2t}}{1-e^{-2t}}$$

Time-evolved measure

$$\mu_t^+(\eta_V) = \int \mu^+(d\sigma)\frac{e^{h_t \sum_{x\in V} \eta_x \sigma_x}}{(2\cosh h_t)^{|V|}}$$

Finite-volume single-site conditional probabilities

$$\mu_t^+(\eta_0|\eta_{V\setminus 0}) = \frac{\int \mu^+(d\sigma)e^{h_t \sum_{x\in V\setminus 0} \eta_x \sigma_x} p_t(\sigma_0, \eta_0)}{\int \mu^+(d\sigma)e^{h_t \sum_{x\in V\setminus 0} \eta_x \sigma_x}}$$

'Quenched Hamiltonian' for fixed **magnetic field configuration** $\eta_{V\setminus 0}$

$$H[\eta_{V\setminus 0}](\sigma) = -\beta \sum_{<x,y>} \sigma_x\sigma_y - h\sum_x \sigma_x - h_t \sum_{x\in V\setminus 0} \eta_x \sigma_x$$

$$\mu_t^+(\eta_0|\eta_{V\setminus 0}) = \int \mu^+[\eta_{V\setminus 0}](d\sigma_0) p_t(\sigma_0, \eta_0)$$

discontinuity of cond. proba. $\leftrightarrow$ phase transition driven by η

Gibbs Regimes:

(0) $\beta << \beta_d$ (high-temperature regime):

Couplings weak $\Rightarrow \mu^+[\eta_{\mathbf{V}\setminus\mathbf{0}}]$ depends only 'locally' (continuously) on η

(show this using Dobrushin-uniqueness)

$\Rightarrow \mu_{\beta,h;t}$ is a Gibbs measure for all $t > 0$

(i) $0 \leq t \leq t_0(\beta, h)$ (small-time regime):

h_t very large $\Rightarrow \sigma_x$ follow essentially $\eta_{\mathbf{x}}$ in the measure $\mu^+[\eta_{\mathbf{V}\setminus\mathbf{0}}]$

(show this using Dobrushin-uniqueness)

$\mu^+[\eta_{\mathbf{V}\setminus\mathbf{0}}]$ depends only 'locally' (continuously) on η

(ii) $h > 0$, large times t:

$h_t << h \Rightarrow \mu^+[\eta_{\mathbf{V}\setminus\mathbf{0}}]$ depends only 'locally' (continuously) on η

Non-Gibbs Regimes:

(iii) low-temperature, large-time regime, $h = 0$:

Claim: $\eta^{\text{spec}} = \mathbf{checkerboard}$ is a bad configuration for μ_t

Indeed: h_t small $\Rightarrow \mu^+[\eta^{\text{spec}}] > \mu^-[\eta^{\text{spec}}]$

$\mu^+[\eta^{\text{spec}}] =$ small perturbation around $\sigma_x \equiv +$

$$\left(\mu^+[\eta^{\text{spec}}_{\mathbf{V}\setminus\mathbf{0}}, \eta^{\pm}_{\mathbf{\Lambda}\setminus\mathbf{V}} = \pm] \approx \mu^{\pm}[\eta^{\text{spec}}] \text{ around } 0 \right)$$

(iv) low-temperature, large-time regime, $h > 0$ not too big:

Find η^{spec} such that $h + h_t \eta^{\text{spec}}_{\mathbf{x}} =$ neutral on average

Ising spins: $\sigma_i \in \{-1, 1\}$, sites $i = 1, \ldots, N$

Finite volume Gibbs measures

$$\mu_{\beta,N}(\sigma_{[1,N]}) := \frac{\exp\left(\frac{\beta}{2N}\left(\sum_{i=1}^{N} \sigma_i\right)^2\right)}{Z_N(\beta)}$$

The dynamics: **Case 1:** Infinite temperature, i.e. p_t independent spin-flip (same as on lattice):

$$\mu_{t,N}(\eta_{[1,N]}) := \sum_{\sigma_{[1,N]}} \mu_N(\sigma_{[1,N]}) \prod_{i=1}^{N} p_t(\sigma_i, \eta_i)$$

$\Rightarrow$ trivial infinite-time limiting measure:

$$\lim_{t\uparrow\infty} \mu_{t,N} = \bigotimes_{i=1}^{N} \frac{1}{2}\left(\delta_+ + \delta_-\right)$$

Spin variables: $\sigma = (\sigma_i)_{i=1,\ldots,N} \in \Omega_0^N$

$L_N(\sigma) = \frac{1}{N}\sum_{i=1}^N \delta_{\sigma_i}$ = empirical distribution

Hamiltonian $\equiv$ energy function $= H(L_N(\sigma))$

$\rho =$ probability measure on Ω_0

Finite volume Gibbs measures:

$$\mu_N(d\sigma_1,\ldots,d\sigma_N) = \frac{1}{Z_N(H,\rho)}\,\mathrm{exp}\Big(-NH\big(L_N(\sigma)\big)\Big)\prod_{i=1}^N \rho(d\sigma_i)$$

μ_N exchangeable $\Rightarrow$ possible limits are mixtures of product measures (de Finetti)

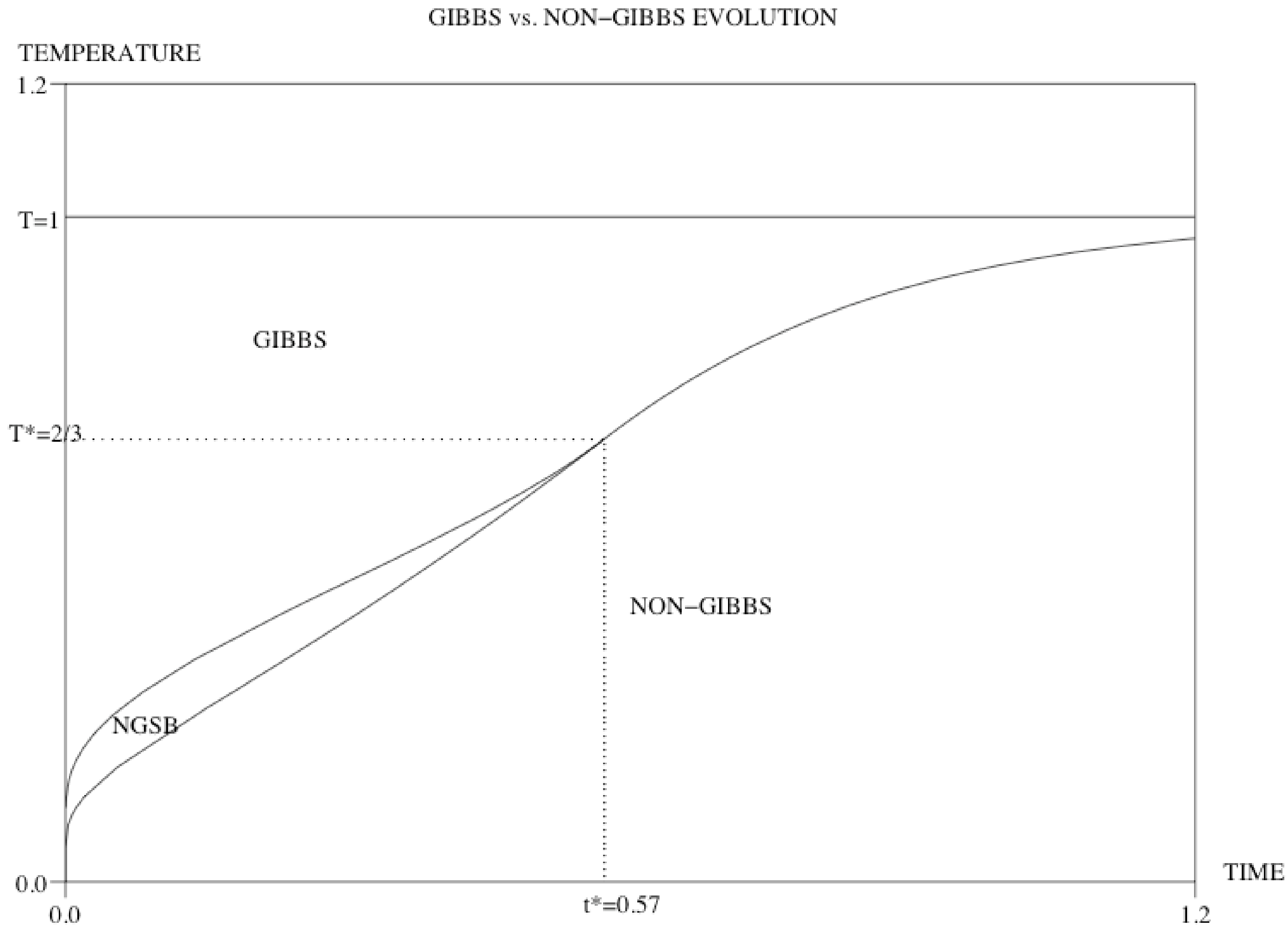
non-trivial mixtures occur means there is a phase-transition

Definition: Call a sequence of exchangeable measures $(\mu_N)_{N\in\mathbb{N}}$ MF-Gibbs if

(i) The limiting conditional probabilities
$\lim_{N\uparrow\infty} \mu_N(d\sigma_1|\sigma_2,\ldots,\sigma_N) = \gamma(d\sigma_1|\nu)$ with $\frac{1}{N}\sum_{i=2}^{N}\delta_{\sigma_i} \to \nu$
exist for all ν

(ii) $\nu \mapsto \gamma(d\sigma_1|\nu)$ is continuous (in weak topology)

Standard-Curie-Weiss: $\gamma(\sigma_1|m) = \frac{e^{\beta m\sigma_1}}{2\cosh(\beta m)}$


GIBBS vs. NON−GIBBS EVOLUTION
TEMPERATURE
1.2
T=1
GIBBS
T*=2/3
NON−GIBBS
NGSB
0.0
0.0
t*=0.57
1.2
TIME

***THEOREM* 41.**

1. *If* $\frac{2}{3} \leq \beta^{-1} < 1$ *the threshold time is given by* $t_0(\beta) = -\frac{1}{4}\log(1-\beta^{-1})$
2. *If* $0 < \beta^{-1} < \frac{2}{3}$ *the boundary-line has the parametrization in terms of* M *of the form*

$$\left\{\begin{pmatrix} t_0(\beta^{-1}) \\ \beta^{-1} \end{pmatrix}, 0 < \beta^{-1} < \frac{2}{3}\right\} = \left\{\begin{pmatrix} t_4(M) \\ \beta_4^{-1}(M) \end{pmatrix}, 0 < M < \infty\right\}$$

with

$$t_4(M) = -\frac{1}{4}\log\frac{2y^3 + M(1-y^2)^2}{2y + M(1-y^2)}$$
$$\beta_4^{-1}(M) = \frac{y(2+My)(1+y^2)}{2M^2y^3 + 2(y+y^3) + M(1+3y^2+y^4)}$$

where $y = y(M) = \tanh(M)$.

effective field depending on time :

$$E := \frac{1}{2}\log\frac{1+e^{-2t}}{1-e^{-2t}}$$

$M = \beta m$ magnetization,

$H = \beta h$ external magnetic field,

α empirical magnetization of conditioning

Potential:

$$\Psi_{\beta^{-1},E,H,\alpha}(M) = \frac{M^2}{2\beta} - \frac{1+\alpha}{2}\log\cosh\Big(M+E+H\Big) - \frac{1-\alpha}{2}\log\cosh\Big(M-E+H\Big)$$

Bifurcation set

$$B = \Big\{(\beta^{-1}, E, H, \alpha) | \exists M \in \mathbb{R}, \\ \partial_M \Psi_{\beta,E,H,\alpha}(M) = 0, \partial_M^2 \Psi_{\beta,E,H,\alpha}(M) = 0\Big\}$$

New phenomenon: Non-Gibbs with symmetry breaking

Proof: Bifurcation analysis: Butterfly unfolding: 1,2,3 minima appearing (related to existence of tricritical point)

Case 2: The dynamics has temperature $(\beta')^{-1}$,
flip each spin σ_i with rate $c(\sigma_i|m)$ where $c(-|m)/c(+|m) = e^{2\beta' m}$
$m = \frac{1}{N-1}\sum_{j=2}^{N} \sigma_j$ is the magnetization of the conditioning

The corresponding time-evolved measure will be called $\mu_{t,\beta,\beta',N}$

$\Rightarrow$ infinite-time limiting measure has inverse temperature β' i.e.

$$\lim_{t\uparrow\infty} \mu_{t,\beta,\beta',N} = \mu_{\beta',N}$$

Jump process for the magnetization:
Increase (lower) the magnetization in a system of size N by $2/N$ with rate

$$\frac{e^{\pm\beta' m}(1 \mp m)}{2(\cosh \beta' m - m \sinh \beta' m)}$$

Consider the single-site conditional probabilities of the time-evolved measure in the volume N, given by

$$\gamma_{\beta,\beta',t,N}(\sigma_1|\widehat{m_N}) := \mu_{\beta,\beta',t,N}(\sigma_1|\sigma_{[2,N]})$$

for any point $\widehat{m}_N = \frac{1}{N-1}\sum_{j=2}^{N}\sigma_j$.

Take $N \uparrow \infty$ at fixed $\widehat{m} = \lim_{N\uparrow\infty}\widehat{m}_N$!

Definition. The time-evolved mean-field model is called **Gibbs** iff

$$\gamma_{\beta,\beta',t}(\sigma_1|\widehat{m}) = \lim_N \gamma_{\beta,\beta',t,N}(\sigma_1|\widehat{m_N})$$

exists for all $\widehat{m}$ and is continuous as a function of $\widehat{m}$.

Picture for low-temperature dynamics

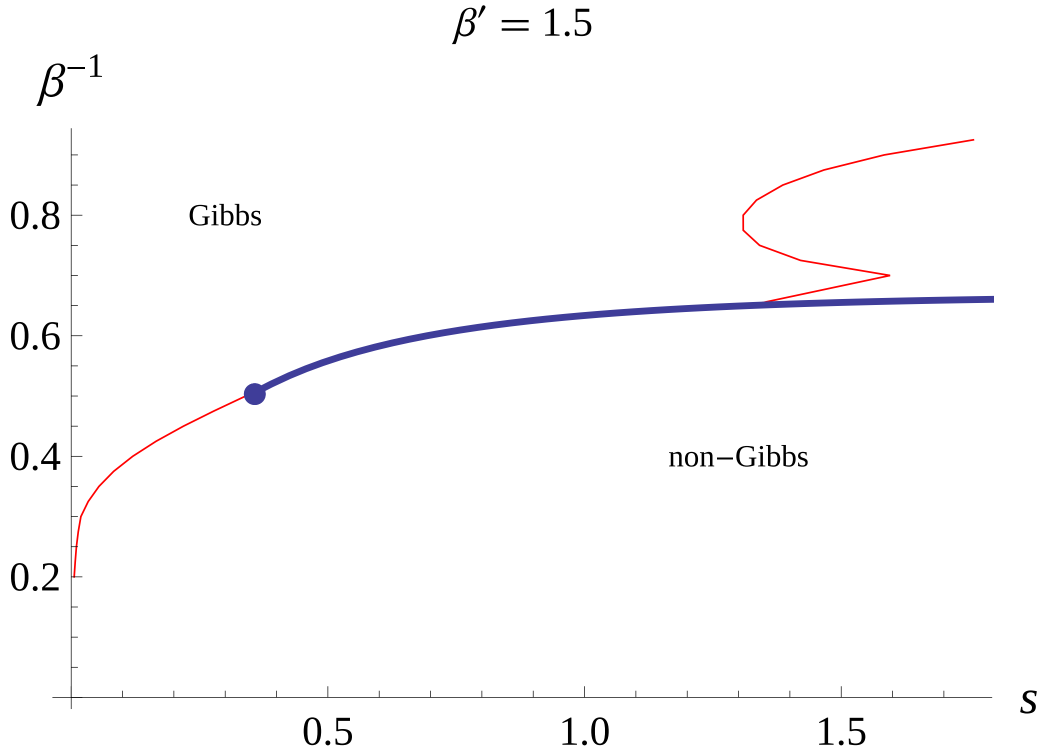


Figure 1: Gibbs and non-Gibbs areas

***THEOREM* 42.** *(Ermolaev, Külske arXiv:1005.0954)*

1. *Initial high temperature, any temperature of the dynamics.*
   If $\beta^{-1} \geq 1$ then the time-evolved model is Gibbs for all $t \geq 0$.

2. *Heating from an initial low temperature, high-temp or a low-temp dynamics.* *If* $0 < \beta^{-1} < \min\{\beta'^{-1}, 1\}$ *there exists a value* $\beta_{SB}^{-1}(\beta')$ *(which is explicitly computable, see below) such that*

   (a) *If* $\beta_{SB}^{-1}(\beta') \leq \beta^{-1}$ *then*

   - *for all* $0 \leq t \leq t_{nGS}(\beta, \beta') := \frac{\ln \frac{\beta' - \beta}{1-\beta}}{4(1-\beta')}$ *the time-evolved model is Gibbs.*
   - *for all* $t > t_{nGS}(\beta, \beta')$ *the model is not Gibbs and the time-evolved conditional probabilities are discontinuous at* $\widehat{m} = 0$ *and continuous at any* $\widehat{m} \neq 0$*.*

   (b) *If* $0 < \beta^{-1} < \beta_{SB}^{-1}(\beta')$ *there exist sharp values* $0 < t_0(\beta, \beta') < t_1(\beta, \beta') < \infty$ *such that*

   - *for all* $0 \leq t \leq t_0(\beta, \beta')$ *the time-evolved model is Gibbs,*
   - *for all* $t_0(\beta, \beta') < t < t_1(\beta, \beta')$ *there exists* $\widehat{m_c} = \widehat{m_c}(\beta, \beta'; t) \in (0, 1)$ *such that the limiting conditional probabilities are discontinuous at the points* $\pm\widehat{m_c}$*, and continuous otherwise,*
   - *for all* $t > t_1(\beta, \beta')$ *the limiting conditional probabilities are discontinuous at* $\widehat{m} = 0$ *and continuous at any* $\widehat{m} \neq 0$*.*

3. *Cooling from initial low temperature. For* $\beta'^{-1} < \beta^{-1} < 1$ *there exists a time threshold* $t_{\text{per}}(\beta, \beta')$ *such that,*
   - *for all* $0 \leq t \leq t_{\text{per}}(\beta, \beta')$ *the time-evolved model is Gibbs.*
   - *for all* $t > t_{\text{per}}(\beta, \beta')$ *the model is not Gibbs and the time-evolved conditional probabilities are discontinuous at non-zero configurations* $\widehat{m}_c$ *(and continuous at* $\widehat{m} = 0$*).*

Moreover:

$$4\beta_{\text{SB}}^3 + 12\beta_{\text{SB}}\beta' - 6\beta_{\text{SB}}^2(1 + \beta') - \beta'(3 + 3\beta' - \beta'^2) = 0$$

In the independent spin-flip case $\beta' = 0$ get back $\beta^{-1} = \frac{2}{3}$.

Sketch of Proof:

Use path-large deviation principle for paths of magnetization $\varphi(s)$

Cost $\sim e^{-N(-\frac{\beta}{2}m^2+I(m))}e^{-N\int_0^t j_{\beta'}(\varphi(s),\dot\varphi(s))ds}$

with a β-dependent Punishment-Term coming from initial measure

Ingredient 1 Denote by $P_{\beta',N}$ the law of the paths $(z_N(s))_{s\in[0,t]}$ of the magnetization for the continuous-time Markov-chain at system size N.
Then the measures $P_{\beta',N}$ satisfy a large deviation principle with rate N and rate function given by the Lagrange functional

$$\varphi \mapsto J_{\beta'}(\varphi) = \int_0^t j_{\beta'}(\varphi(s),\dot\varphi(s))ds$$

with Lagrange density $j_{\beta'}(m,v)$ given by

$$
\begin{aligned}
j_{\beta'}(m,v) = \frac{1}{2}\Bigg\{ & 2 \\
& - \sqrt{\frac{e^{4\beta' m}(-1+m)^2 v^2 + (1+m)^2 v^2 - 2e^{2\beta' m}\left(-1+m^2\right)\left(8+v^2\right)}{\left(1-e^{2\beta' m}(-1+m)+m\right)^2}} \\
& + v \log\left[\frac{e^{-2\beta' m}\left(-1+e^{2\beta' m}(-1+m)-m\right)}{4(-1+m)}\right] \\
& + v \log\left[v + \sqrt{\frac{e^{4\beta' m}(-1+m)^2 v^2 + (1+m)^2 v^2 - 2e^{2\beta' m}\left(-1+m^2\right)\left(8+v^2\right)}{\left(1-e^{2\beta' m}(-1+m)+m\right)^2}}\right]\Bigg\}
\end{aligned}
$$

For the special important case of non-interacting dynamics $\beta' = 0$ we have

$$
j_0(m,v) = \frac{1}{2}\left(2 - \sqrt{4 - 4m^2 + v^2} + v \log\left[\frac{v + \sqrt{4 - 4m^2 + v^2}}{2 - 2m}\right]\right)
$$

Proof: Approximation by Compound Poisson process

or see Feng-Kurtz-Book: LD for stochastic processes

Consequence: Suppose the total cost function

$$-\frac{\beta}{2}m^2 + I(m) + \inf_{\varphi(0)=m,\varphi(t)=m'} \int_0^t j_{\beta'}(\varphi(s),\dot{\varphi}(s))ds - \text{Const}$$

has a unique minimizer $s \mapsto m^*(s;m',t)$ then

$$\gamma_{\beta,\beta',t}(\eta_1|m') = \frac{\sum_{\sigma_1=\pm 1} e^{-\sigma_1\beta m^*(0;m',t)} p_t(\sigma_1,\eta_1;m',t)}{\sum_{\sigma_1,\tilde{\eta}_1=\pm 1} e^{-\sigma_1\beta m^*(0;m',t)} p_t(\sigma_1,\tilde{\eta}_1;m',t)}$$

$p_s(\sigma_1,\eta_1;m',t) =$ transition probability for Markov jump process on $\{-1,1\}$ with the time-dependent generator

$$L(s;m',t)f(\sigma_1) = c(\sigma_1, m^*(s;m',t))(f(-\sigma_1) - f(\sigma_1))$$

Constrained variational problem over paths φ with $\varphi(t) = m'$

Necessary condition for an extremum:

Euler-Lagrange equation and free left-end condition of the form

$$\begin{aligned}
\frac{d}{ds} j_{\dot\varphi}(\varphi(s), \dot\varphi(s)) - j_{\varphi}(\varphi(s), \dot\varphi(s)) &= 0 \text{ for all } s \in [0, t] \\
j_{\dot\varphi}(\varphi(s), \dot\varphi(s)) + H_{\varphi}(\varphi(s)) + I_{\varphi}(\varphi(s))|_{s=0} &= 0 \\
\varphi(t) &= m'
\end{aligned}$$

First integral:

$$j(\varphi(s), \dot{\varphi}(s)) - \dot{\varphi}(s) j_{\dot{\varphi}}(\varphi(s), \dot{\varphi}(s)) = C$$

equivalent to

$$\dot{m} = \pm \sqrt{C + \frac{16 e^{2\beta' m}(m^2 - 1)}{\left(1 - e^{2\beta' m}(m - 1) + m\right)^2}}$$

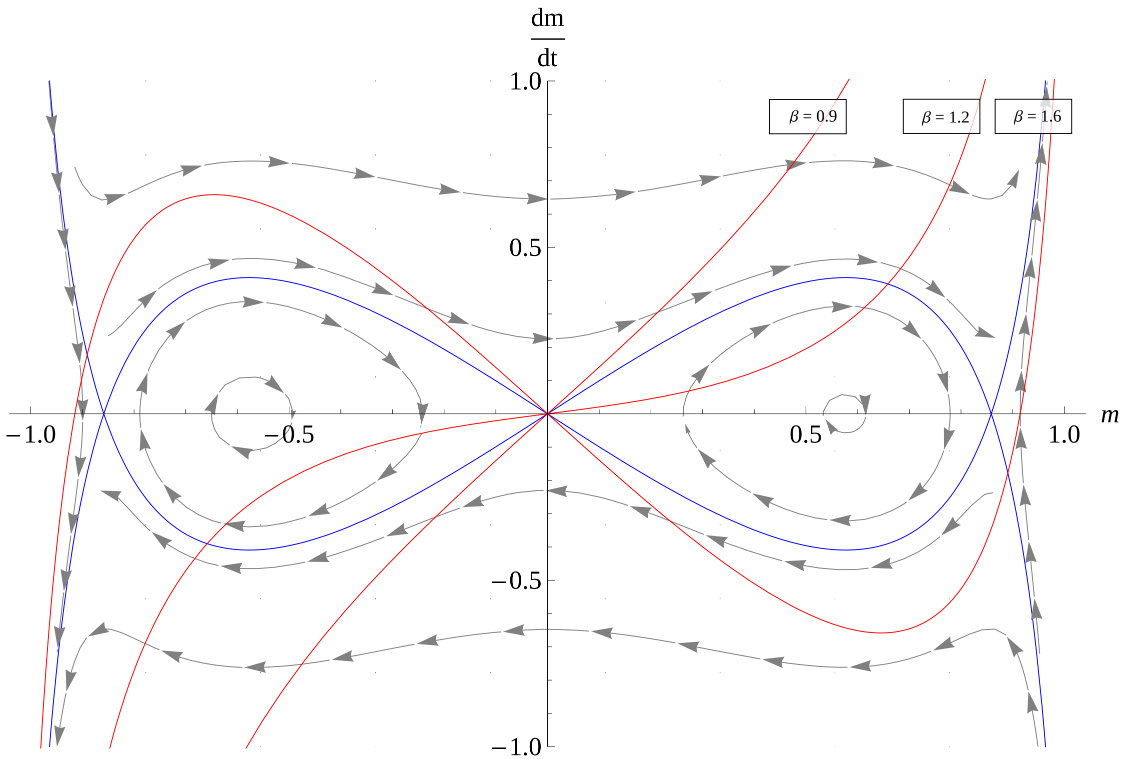


Figure 2: Phase portrait with level curves and *ACC*, $\beta' = \frac{3}{2}$

Denote the allowed initial configurations curve by $m \mapsto \dot{m} = g(m)$

Call $(\beta, \beta', t, m_{\text{pb}})$ *pre-bad* iff there exists $m_{0,1} \neq m_{0,2}$ such that

$$m(t; m_{0,1}, g(m_{0,1})) = m(t; m_{0,2}, g(m_{0,2})) = m_{\text{pb}}$$

Call $(\beta, \beta', t, m_{\text{bad}})$ *bad* iff the paths started at $m_{0,1} \neq m_{0,2}$ are both minimizers for the total cost

Absence of pre-bad points implies the model is Gibbs

Existence of bad points (multiple histories) implies (generically) that the model in non-Gibbs

Region 2a) of non-symmetry-breaking non-Gibbsianness

$$\frac{2}{3} = \beta_{\mathsf{SB}}^{-1}(\beta' = 0) \le \beta^{-1} < 1$$

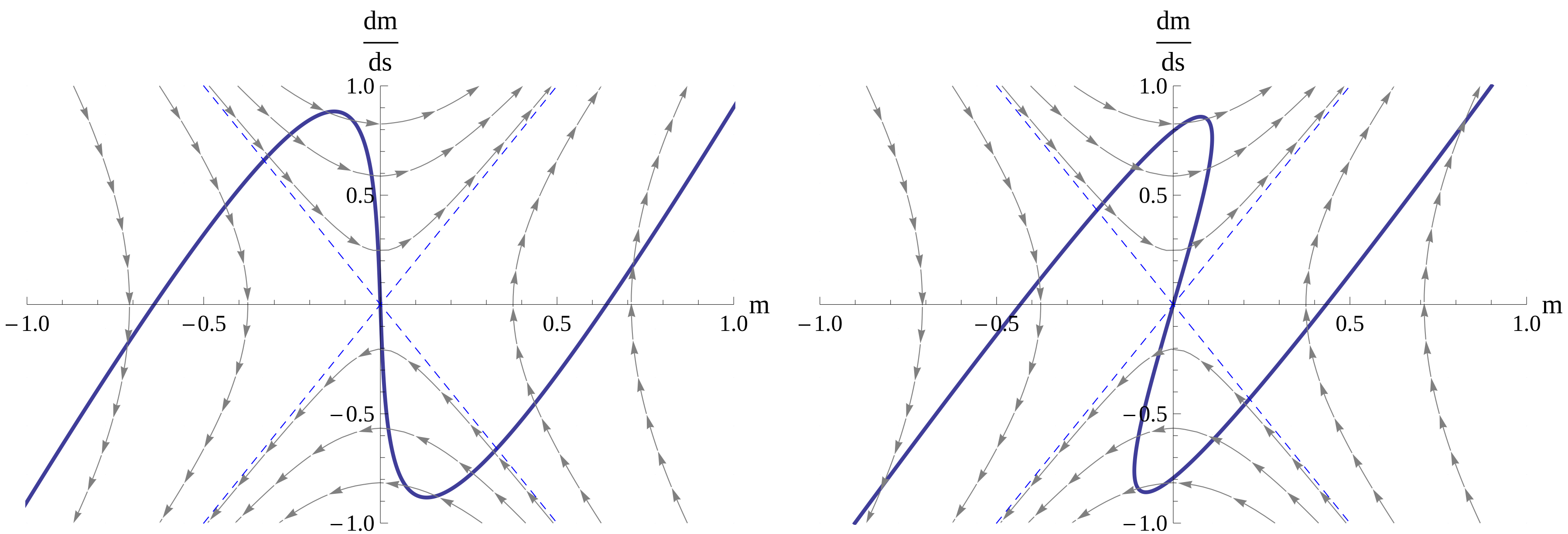


Figure 3: Non-symmetry-breaking mechanism, $\beta' = 0, \beta^{-1} = 0.8$

Region 2b) of symmetry-breaking non-Gibbsianness i.e. $\beta^{-1} < \beta_{\mathrm{SB}}^{-1}(\beta' = 0)$

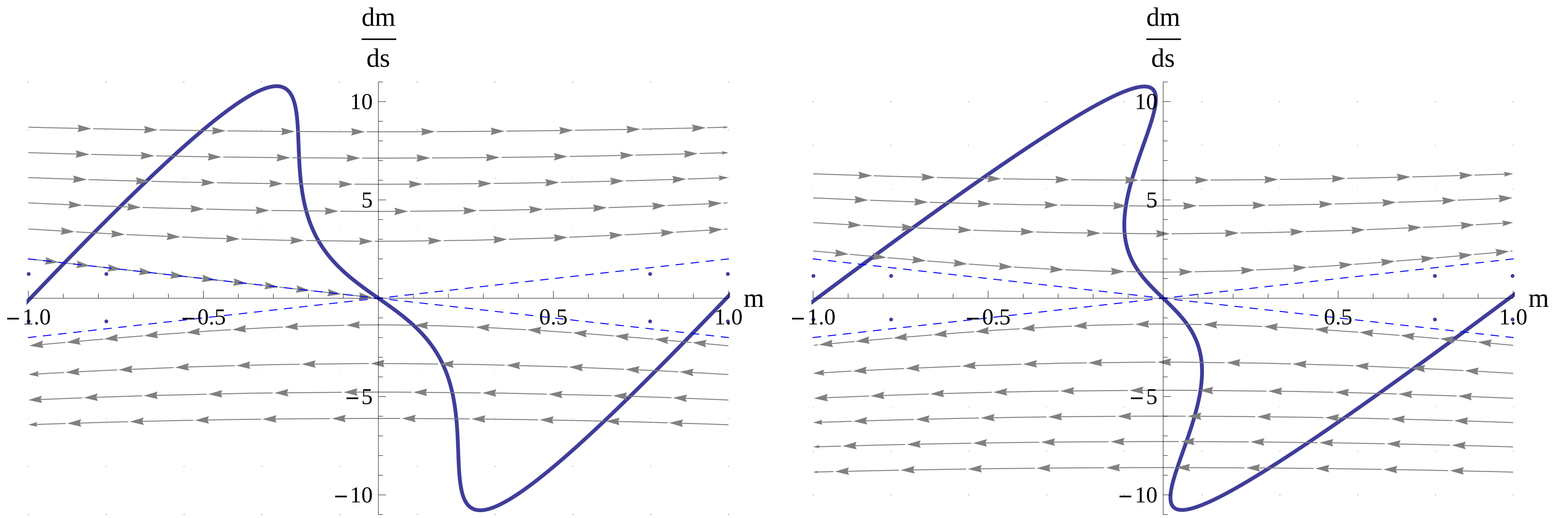


Figure 4: Symmetry-breaking mechanism, $\beta' = 0, \beta^{-1} = 0.4$

Region 3 cooling from initial low-temperature. $\frac{2}{3} = \beta'^{-1} < \beta^{-1} = 0.85 < 1$

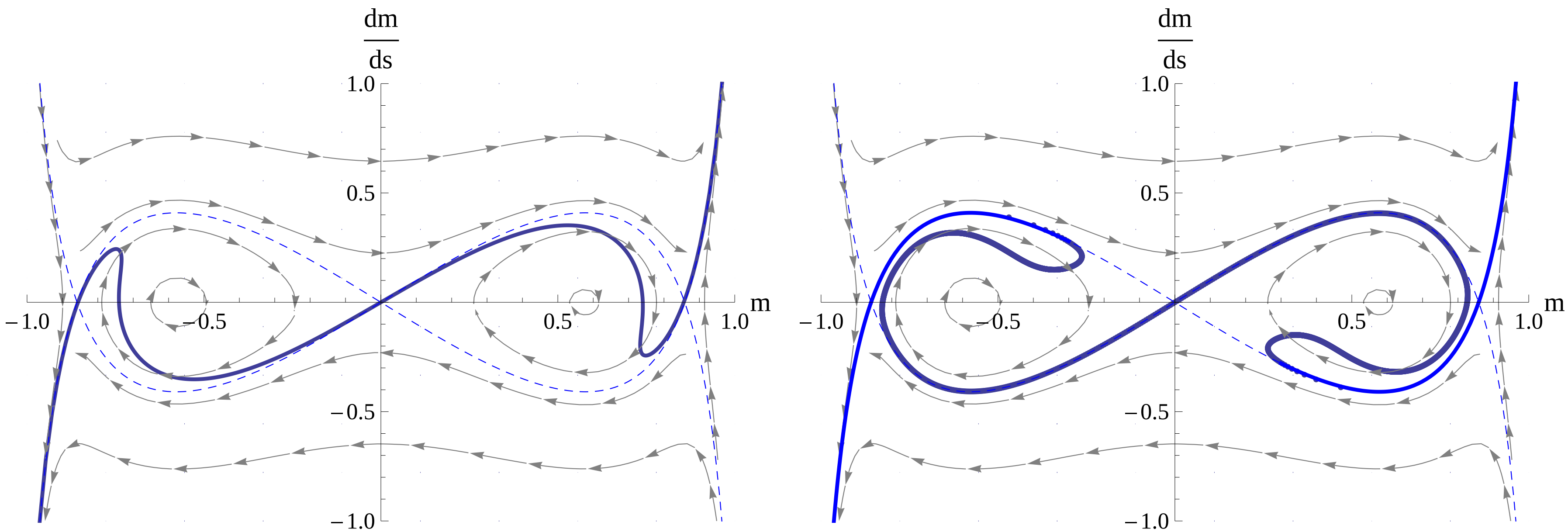
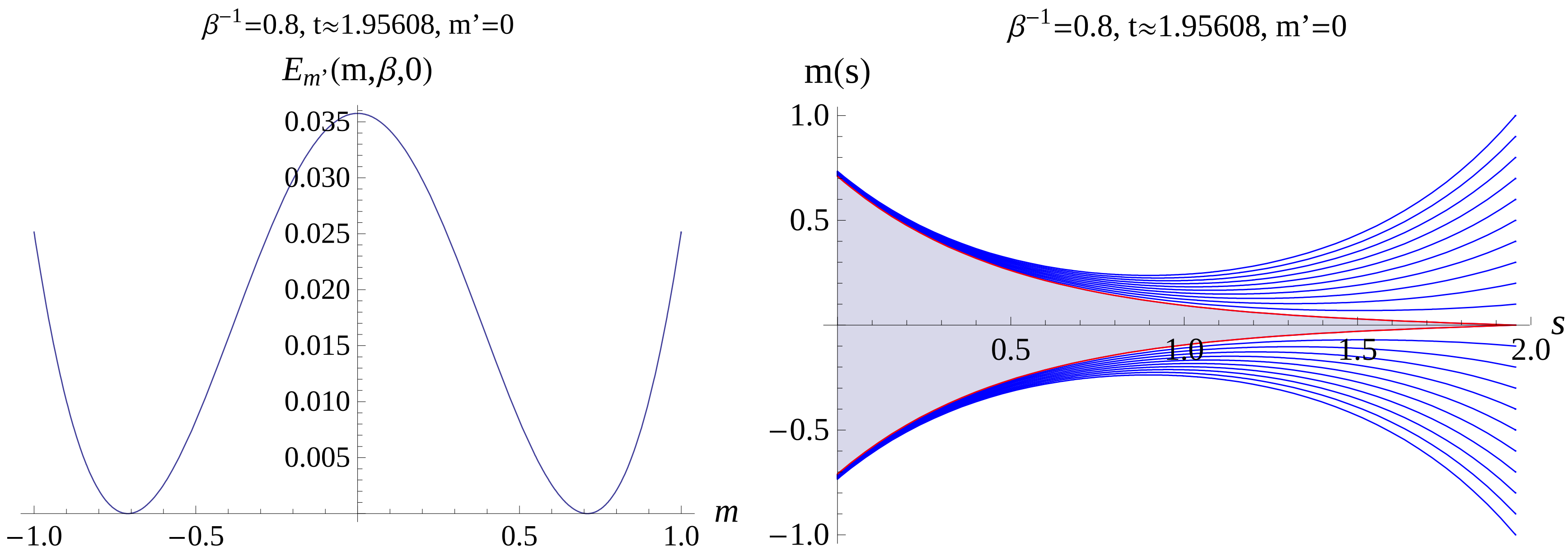


Figure 5: Non-Gibbsianness by periodicity, $\beta'^{-1} = \frac{2}{3}, \beta^{-1} = 0.85$



Cost functional

Corresponding history curves

Figure 6: Symmetric forbidden regions

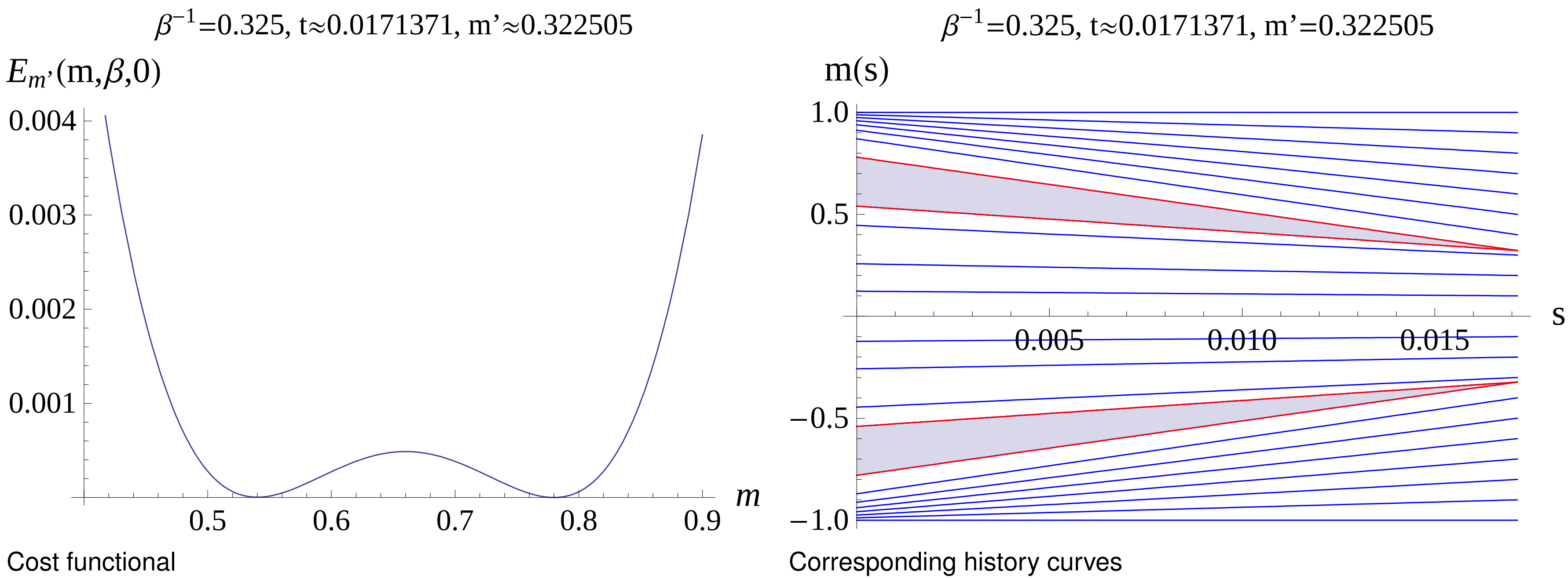


Figure 7: Non-symmetric forbidden region

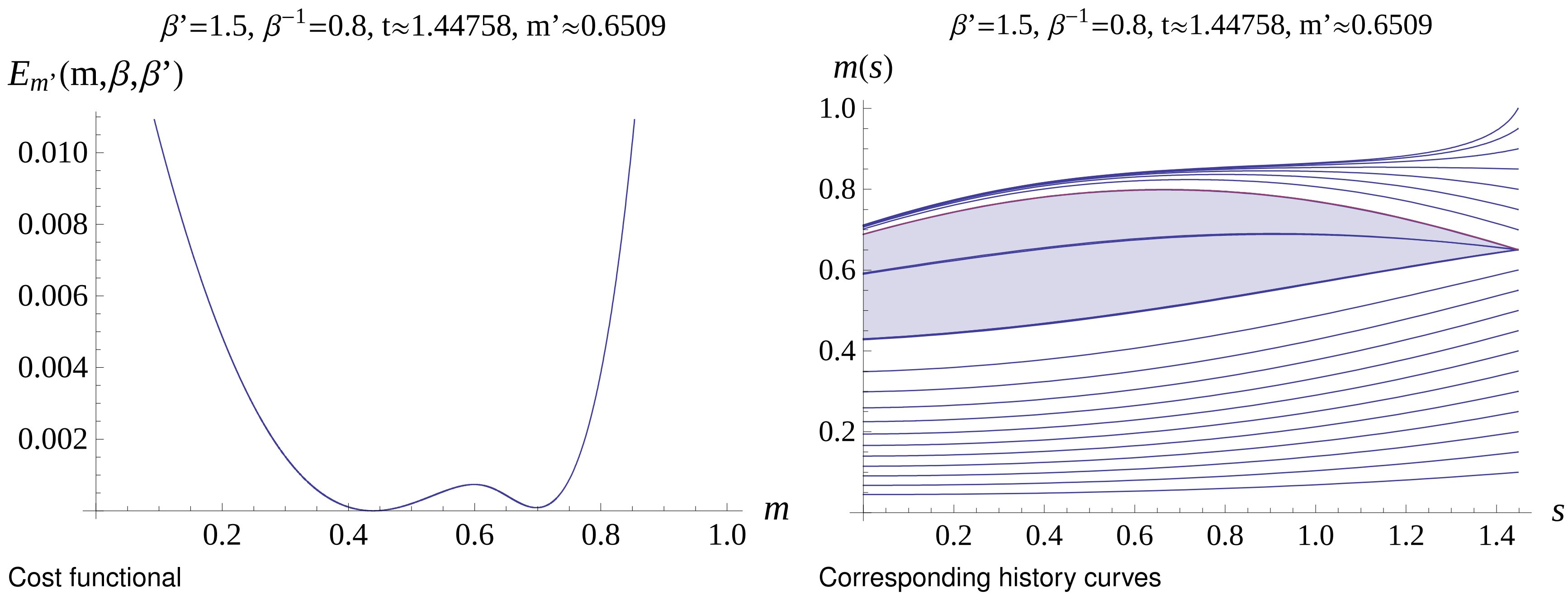


Figure 8: Forbidden region for $\beta' = \frac{3}{2}$

Multiple histories lead to discontinuous conditional probabilities

Can there be infinitely many branches of bad configurations?

What can we learn from that for the lattice?

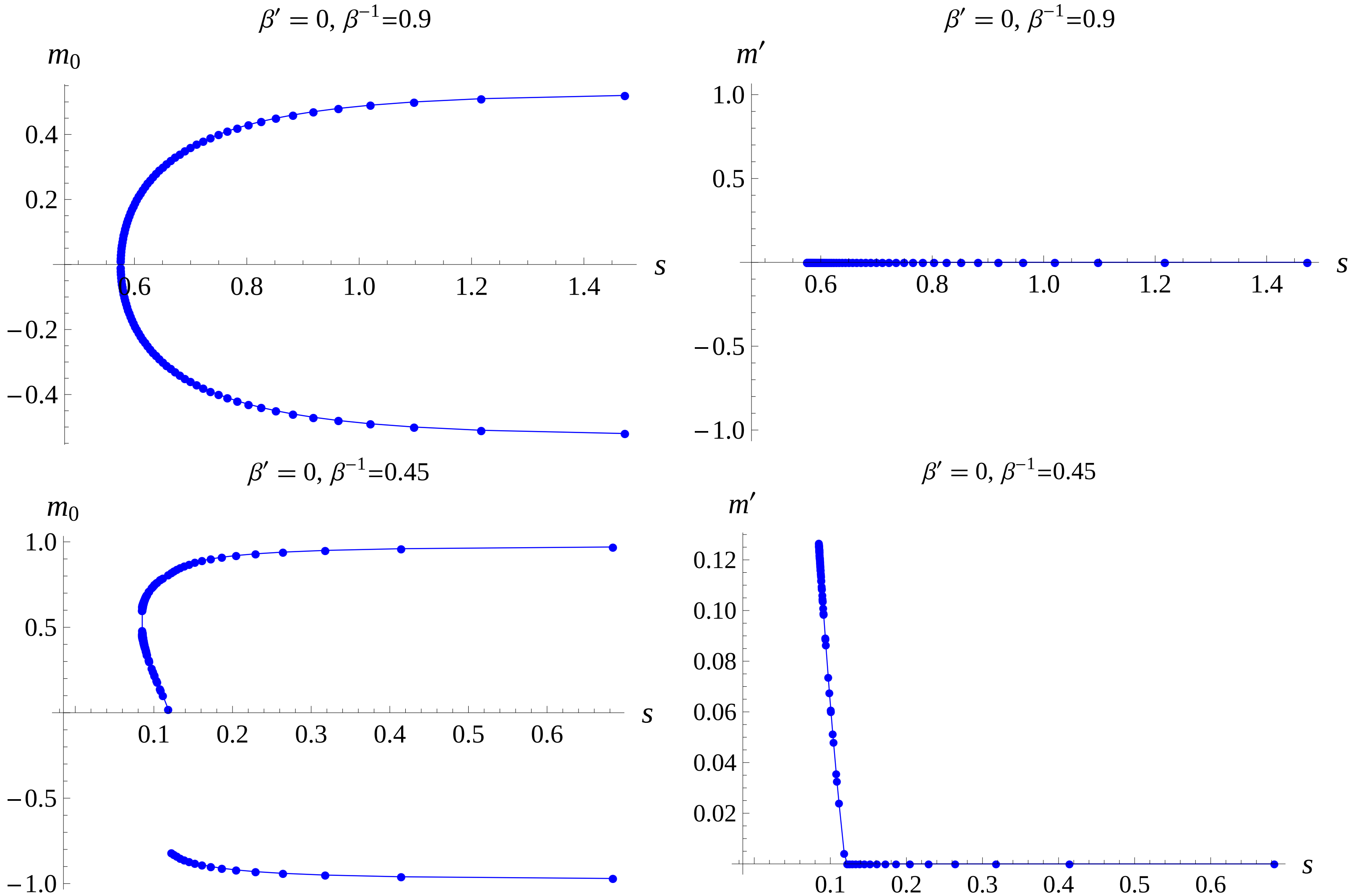


Figure 9: Bad configurations as function of time (right) and initial points of trajectories (left) $\beta' = 0$

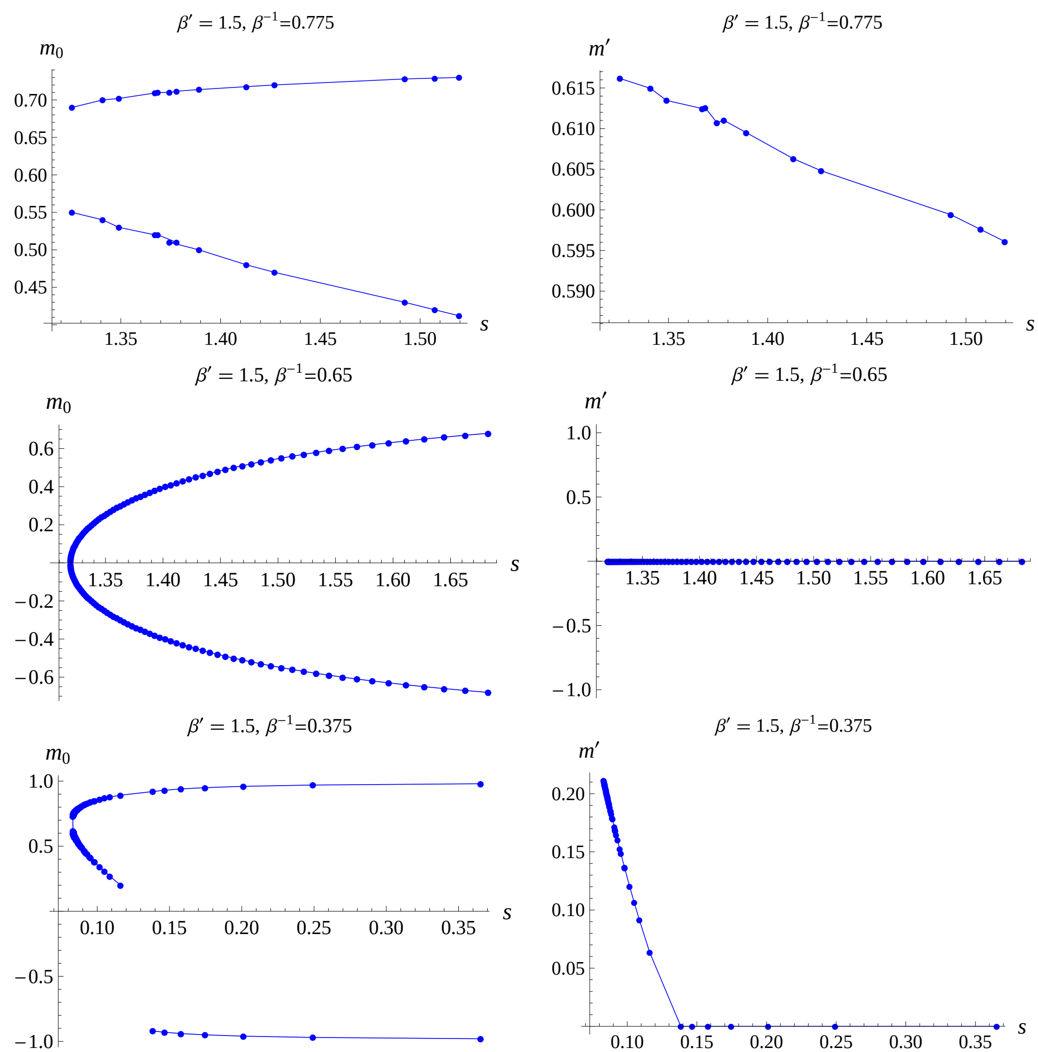


Figure 10: Bad configurations as function of time (right) and initial points of trajectories (left) - low-temperature dynamics $\beta' = 1.5$