One of the key ingredients in the proof of the theorem is the Calderon-Zygmund inequality which we now prove.

1.4 The Calderon-Zygmund inequality

Let $\Omega \subseteq \mathbb{C}$ be an open set, $u \in C^\infty(\Omega)$. Then

$$\begin{aligned} \langle \bar\partial u, f \rangle_2 &= \int_\Omega \bar\partial u \cdot \bar f \, dx\, dy \\ &= -\int_\Omega u \cdot \bar\partial \bar f \, dx\, dy \\ &= -\int_\Omega u \cdot \overline{(\partial f)} \, dx\, dy \\ &= \langle u, -\partial f \rangle_2 , \quad \forall f \in C_c^\infty(\Omega). \end{aligned}$$

Def. $u \in L^1_{loc}(\Omega)$ is weak solution of $\bar\partial u = g$ if

$$\langle u, \partial f \rangle_{L^2} = -\langle g, f \rangle_{L^2}, \quad \forall f \in C_c^\infty(\Omega). \quad \mathrm{I}$$

Similarly, $u \in L^1_{loc}(\Omega)$ is weak solution of $\Delta u = g$ if

$$\langle u, \Delta f \rangle_{L^2} = \langle g, f \rangle_{L^2}, \quad \forall f \in C_c^\infty(\Omega).$$

Rmk. The fundamental solution of $\bar\partial u = 0$ is

$$N(z) = \frac{1}{\pi z}, \qquad (\bar\partial N = \delta \text{ Dirac delta})$$

whereas the fundamental solution of $\Delta u = 0$ is

$$K(z) = \frac{1}{2\pi} \cdot \ln|z|. \qquad (\Delta K = \delta \text{ Dirac delta})$$

In particular,

$$N(z) = 4 \cdot \frac{\partial K}{\partial z}(z),$$

$\langle N, \partial f \rangle_2 = -f(0)$ $\forall f \in C_c^\infty(\mathbb{C})$

and $N, K \in L^1_{loc}(\mathbb{C})$. Therefore

$$\left.\begin{aligned} \partial_x (K * f) &= \partial_x K * f \\ \partial_y (K * f) &= \partial_y K * f \end{aligned}\right\} \quad \forall f \in C_c^\infty(\mathbb{C}).$$

Such identities do not hold instead for the second derivatives.

Poisson's identity

- $K * \Delta u = u \,, \quad \partial_x K * \Delta u = \partial_x u \,,$
  $\partial_y K * \Delta u = \partial_y u \,, \quad \forall\, u \in C_c^2(\mathbb{C}) .$

- $\Delta(K * f) = f , \quad \Delta(\partial_x K * f) = \partial_x f ,$
  $\Delta(\partial_y K * f) = \partial_y f , \quad \forall\, f \in C_c^\infty(\mathbb{C}) .$

<u>Lemma</u> $u, f \in L^1(\mathbb{C})$ with compact support. Then:

(i) u weak solution of $\Delta u = f \iff u = K * f$.
(ii) u weak solution of $\Delta u = \partial_x f \iff u = \partial_x K * f$.

<u>proof.</u>

(i) $\boxed{\Leftarrow}$ For $\varphi \in C_c^\infty(\mathbb{C})$ here

$$\begin{aligned}
\langle u, \Delta\varphi \rangle_2 &= \int u \cdot \overline{\Delta\varphi} \\
&= \int (K * f) \cdot \overline{\Delta\varphi} \\
&= \int f \cdot (K * \overline{\Delta\varphi}) \\
&= \int f \cdot \overline{\varphi} \\
&= \langle f, \varphi \rangle_2 . \quad \checkmark
\end{aligned}$$

$\boxed{\Rightarrow}$ Let u be a weak solution of $\Delta u = f$, and let ρ_ε be mollifiers tending to δ for $\varepsilon \downarrow 0$. Then

$$\begin{aligned}
\langle \Delta\rho_\varepsilon * u, \varphi \rangle_2 &= \int (\Delta\rho_\varepsilon * u)\, \overline{\varphi} \\
&= \int u \cdot (\Delta\rho_\varepsilon * \overline{\varphi}) \\
&= \langle u, \Delta(\rho_\varepsilon * \varphi) \rangle_2 \\
&= \langle f, \rho_\varepsilon * \varphi \rangle_2
\end{aligned}$$

$= \langle \rho_\varepsilon * f, \varphi \rangle_2 \quad , \quad \forall \varphi \in C_c^\infty(\mathbb{C}),$

i.e. $\Delta(\rho_\varepsilon * u) = \rho_\varepsilon * f$. It follows that

$$\Delta(\rho_\varepsilon * u - K * (\rho_\varepsilon * f)) = 0,$$

i.e. $\rho_\varepsilon * u - K * (\rho_\varepsilon * f)$ bounded infinitesimal harmonic function $\Longrightarrow$ $\rho_\varepsilon * u = K * (\rho_\varepsilon * f)$. The claim follows taking the limit $\varepsilon \downarrow 0$.

(ii) analogous. ■

Rmk. Since the second derivatives of K are not in L^1_{loc} can find $f : \mathbb{C} \to \mathbb{R}$ continuous s.t. $K * f \notin C^2$. For such an f $\nexists$ classical solutions of $\Delta u = f$.

Lemma (Weyl) Every weak solution $u \in L^1_{loc}(\Omega)$ of $\bar\partial u = 0$ resp. $\Delta u = 0$ is holomorphic resp. harmonic.

proof.

Have "$\Delta(\rho_\varepsilon * u) = \rho_\varepsilon * \Delta u = 0$", that is $\rho_\varepsilon * u$ harmonic on

$$\Omega_\varepsilon := \{ x \in \Omega \mid B_\varepsilon(x) \subseteq \Omega \}$$

$\Longrightarrow$ u_ε satisfies the mean value property. Since $u_\varepsilon \to u$ in $L^1_{loc}(\Omega)$, u also satisfies the mean value property, thus is harmonic. ■

Lemma Let $f : \mathbb{C} \to \mathbb{R}$ measurable, $t > 0$. Set

$$\mu(t, f) := \lambda(\{ x \in \mathbb{C} \mid |f(x)| > t \}),$$

where λ is the Lebesgue measure. Then, if $f \in L^p(\mathbb{C})$ with $p < +\infty$, we have

$$t^p \cdot \mu(t, f) \leq \int |f|^p \, dx = p \cdot \int_0^{+\infty} s^{p-1} \cdot \mu(s, f) \, ds.$$

II

In particular
$$\mu(t, f+g) \leq \mu(\tfrac{t}{2}, f) + \mu(\tfrac{t}{2}, g).$$

proof.

Integrate the function
$$F(x,t) := \begin{cases} p\,t^{p-1} & 0 \leq t \leq |f(x)|, \\ 0 & \text{otherwise}. \end{cases}$$

Lemma (Marcinkiewicz) Let $T \in L(L^2(\mathbb{C}))$ such that
$$\mu(t, Tf) \leq C \cdot \frac{\|f\|_1}{t}, \quad \forall\, t > 0,$$
$\forall\, f \in L^1(\mathbb{C}) \cap L^2(\mathbb{C})$. Then, $\forall\, 1 < p < 2$, $\exists\, c = c(\|T\|, C, p) > 0$ such that
$$\|Tf\|_p \leq c \cdot \|f\|_p, \quad \forall\, f \in L^1(\mathbb{C}) \cap L^2(\mathbb{C}).$$
In particular, $T|_{L^1(\mathbb{C}) \cap L^2(\mathbb{C})}$ extends (uniquely) to $T_p \in L(L^p(\mathbb{C}))$.

proof.

For $f \in L^1(\mathbb{C}) \cap L^2(\mathbb{C})$ define $f_t, g_t : \mathbb{C} \to \mathbb{R}$ by
$$f_t^+(x) := \begin{cases} f(x) & |f(x)| > t, \\ 0 & \text{otherwise}, \end{cases}$$

$$f_t^-(x) := \begin{cases} f(x) & |f(x)| \leq t, \\ 0 & \text{otherwise}. \end{cases}$$

Clearly $f = f_t^+ + f_t^-$, $\forall\, t > 0$. By the preceeding lemma
$$\begin{aligned} \mu(t, Tf) &\leq \mu(\tfrac{t}{2}, Tf_t^+) + \mu(\tfrac{t}{2}, Tf_t^-) \\ &\overset{\substack{\text{assumption} \\ \text{+ lemma} \\ \text{above with } p=2}}{\leq} C \frac{2 \cdot \|f_t^+\|_1}{t} + \left(\frac{2}{t}\right)^2 \cdot \int |Tf_t^-(x)|^2\, dx \\ &\leq \frac{2C}{t} \|f_t^+\|_1 + \frac{4\|T\|^2}{t^2} \cdot \|f_t^-\|_2^2. \end{aligned}$$

We deduce for $1 < p < 2$:

$$\|Tf\|_p^p = \int |Tf(x)|^p \, dx$$

$$= p \cdot \int_0^{+\infty} t^{p-1} \cdot \mu(t, Tf) \, dt$$

$$\leq 2p\, C_1 \cdot \underbrace{\int_0^{+\infty} t^{p-2} \cdot \|f_t^+\|_1 \, dt}_{(*)} + 4p \cdot \|T\|^2 \cdot \underbrace{\int_0^{+\infty} t^{p-3} \cdot \|f_t^-\|_2^2 \, dt}_{(**)}$$

with

$$(*) = \int_0^{+\infty} t^{p-2} \left(\int_{|f|>t} |f(x)| \, dx \right) \overset{\text{Fubini}}{=} \int |f(x)| \cdot \left(\int_0^{|f(x)|} t^{p-2} \, dt \right) dx$$

$$= \frac{1}{p-1} \cdot \int |f(x)|^p \, dx$$

and similarly

$$(**) = \frac{1}{2-p} \cdot \int |f(x)|^p \, dx .$$

Therefore, we conclude

$$\|Tf\|_p^p \leq \left(\frac{2p\, C_1}{p-1} + \frac{4p \|T\|^2}{2-p} \right) \cdot \|f\|_p^p . \qquad \blacksquare$$

I' • $u \in C_c^2(\mathbb{C}) \Rightarrow u = N * \bar\partial u$

• $f \in C_c^\infty(\mathbb{C}) \Rightarrow f = \bar\partial(N * f)$

<u>Theorem</u> (Calderon-Zygmund inequality)

For every $p \in (1, +\infty)$, $\exists$ $c = c(p) > 0$ s.t.

$$(+) \qquad \|\nabla(N * u)\|_p \leq c \cdot \|u\|_p , \quad \forall u \in C_c^\infty(\mathbb{C}).$$

As a corollary,

$$(++) \qquad \|\nabla u\|_p \leq c \cdot \|\bar\partial u\|_p , \quad \forall u \in C_c^\infty(\mathbb{C}).$$

<u>Rmk.</u> (++) follows from (+) applied to $\bar\partial u$. Indeed

$$\|\nabla u\|_p = \|\nabla(N * \bar\partial u)\|_p \overset{(+)}{\leq} c \cdot \|\bar\partial u\|_p .$$

<u>proof.</u>

<u>Step 1</u> (+) is true for $p = 2$.

For $f \in C_c^\infty(\mathbb{C})$ set $u = N * f$ and observe that

$$\int_{B_R} |\nabla u|^2 = -\int_{B_R} u \cdot \Delta u + \int_{\partial B_R} u \cdot \frac{\partial u}{\partial \nu}$$

with

$$\Delta u = \partial\bar\partial u = \partial\bar\partial(N * f) = \bar\partial N * \partial f$$
$$= 4\,\bar\partial\partial K * \partial f = 4\cdot \Delta K * \partial f$$
$$= 4\delta * \partial f = 4\partial f.$$

Also

$$|u(z)| + |\nabla u(z)| \leq \frac{c}{|z|} \quad \text{for } |z| > C$$

and hence

$$\int_{\partial B_R} u \cdot \frac{\partial u}{\partial \nu} \xrightarrow[R\to+\infty]{} 0.$$

Thus

$$\int |\nabla u|^2 = -\int u\cdot \Delta u$$
$$= -\int u \cdot 4\partial f$$
$$= \int 4\bar\partial u \cdot f$$
$$\overset{\text{Cauchy-Schwarz}}{\leq} 4\,\|\bar\partial u\|_2 \cdot \|f\|_2$$
$$\leq 4\,\|\nabla u\|_2 \cdot \|f\|_2$$

$$\Longrightarrow \|\nabla u\|_2 \leq 4\cdot\|f\|_2. \quad ✓$$

III

Step 2 By Step 1, the operator

$$T: f \mapsto \partial_x(N * f), \quad f \in C_c^\infty(\mathbb{C}),$$

extends to a bounded operator on $L^2(\mathbb{C})$. We claim that T satisfies the assumptions of Marcinkiewicz's lemma, i.e. $\exists\ C_1 > 0$ s.t.

digression on Marcinkiewicz

$$\mu(t, Tf) \leq \frac{C_1}{t}\cdot\|f\|_1, \quad \forall\ f \in L^1(\mathbb{C}) \cap L^2(\mathbb{C}).$$

We take Step 2 as a black box and complete the proof.

→ IV Step 2 : Show that T satisfies Marcinkiewicz

Step 3 In case $1 < p < 2$, the claim follows applying Marcinkiewicz's lemma to $f \mapsto \partial_x(N * f)$ and $f \mapsto \partial_y(N * f)$. For $p > 2$ let $q \in (1,2)$ be the conjugated exponent. Then $\forall f, g \in C_c^\infty(\mathbb{C})$:

$$\langle g, \partial_x(N * f) \rangle_2 = \langle \partial_x(N * g), f \rangle_2$$

$$\leq c \cdot \| f \|_p \cdot \| \partial_x (N * g) \|_q$$

$$\leq c \cdot \| f \|_p \cdot \| g \|_q$$

$$\Longrightarrow \| \partial_x (N * f) \|_p \leq c \cdot \| f \|_p .$$

Theorem (Calderon-Zygmund for Δ)
For every $p \in (1, +\infty)$, $\exists$ $c = c(p, m) > 0$ s.t.

$$\| \partial_{kj} (K * f) \|_p \leq c \cdot \| f \|_p , \quad \forall f \in C_c^\infty(\mathbb{R}^m),$$

$\forall$ $k, j = 1, \dots, m$. As a corollary

$$\| \partial_{kj} u \|_p \leq c \cdot \| \Delta u \|_p , \quad \forall u \in C_c^\infty(\mathbb{R}^m).$$

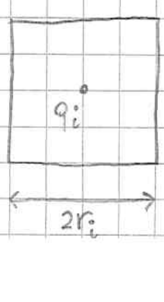
proof of Step 2

Claim 1 $\exists$ $c > 0$ s.t., if $Q = \bigcup_{i=1}^{+\infty} Q_i$ union of squares with pairwise disjoint interior, then $\forall$ $h \in L^1(\mathbb{C})$ with support in Q and such that $\int_{Q_i} h = 0$, $\forall i$, we have

$$\nabla \qquad \mu(t, Th) \leq c \cdot \left(\mathrm{vol}(Q) + \frac{1}{t} \| h \|_1 \right).$$

Set $h_i := h|_{Q_i} \in L^1(\mathbb{C})$, and denote with q_i the center of Q_i and with $2r_i$ the length of a side of Q_i.

Then



$$d(y, q_i) \leq \sqrt{2} \cdot r_i , \quad \forall y \in Q_i ,$$

and for $x \notin Q_i$ we have

$$
\begin{aligned}
|Th_i(x)| &= \left| \partial_x (N * h_i)(x) \right| \\
&= \left| \int_{\mathbb{R}^2} \partial_x N(x-y) \cdot h_i(y)\, dy \right| \\
&= \left| \int_{Q_i} \partial_x N(x-y) \cdot h_i(y)\, dy \right| \\
\int_{Q_i} h = 0 \curvearrowright \quad &= \left| \int_{Q_i} \Big(\partial_x N(x-y) - \partial_x N(x-q_i) \Big) \cdot h_i(y)\, dy \right| \\
&\leq \int_{Q_i} \left| \partial_x N(x-y) - \partial_x N(x-q_i) \right| \cdot |h_i(y)|\, dy \\
&\leq \|h_i\|_1 \cdot \max_{y \in Q_i} \left| \partial_x N(x-y) - \partial_x N(x-q_i) \right| \\
&\leq \|h_i\|_1 \cdot \sqrt{2} r_i \cdot \max_{y \in Q_i} \left| \nabla \partial_x N(x-y) \right| \\
&\leq c_1 \cdot r_i \|h_i\|_1 \cdot \max_{y \in Q_i} |x-y|^{-3} \\
&\leq c_1 \cdot r_i \|h_i\|_1 \cdot d(x, Q_i)^{-3}.
\end{aligned}
$$

Setting

$$B_i := \{ x \mid |x - q_i| < 2\sqrt{2} r_i \}$$

we obtain for $x \in \mathbb{C} \setminus B_i$:

$$d(x, Q_i) \geq |x - q_i| - \sqrt{2} \cdot r_i$$

and hence

$$
\begin{aligned}
&\int_{\mathbb{C} \setminus B_i} |Th_i(x)|\, dx \\
&\leq c_1 \cdot r_i \cdot \|h_i\|_1 \cdot \int_{|x| > 2\sqrt{2} r_i} \frac{dx}{(|x| - \sqrt{2} r_i)^3} \\
\text{polar coordinates} \curvearrowright \quad &\leq C_2 \cdot \|h_i\|_1 \\
&= C_2 \cdot \|h\|_{L^1(Q_i)}.
\end{aligned}
$$

Therefore, with $B := \bigcup_{i=1}^{+\infty} B_i$ we obtain

$$\int_{\mathbb{C} \setminus B} |Th(x)|\, dx \leq C_2 \cdot \|h\|_1 ,$$

and since

$$\mathrm{vol}(B) \leq c_3 \cdot \mathrm{vol}(Q),$$

we conclude

$$\begin{aligned} t \cdot \mu(t, Th) &\leq t \cdot \mathrm{vol}(B) + t \cdot \lambda\left(\{x \in \mathbb{C} \setminus B \,/\, |Th(x)| > t\}\right) \\ &\leq t \cdot \mathrm{vol}(B) + \int_{\mathbb{C} \setminus B} |Th(x)|\, dx \\ &\leq c_4 \left(t \cdot \mathrm{vol}(B) + \|h\|_1 \right). \end{aligned}$$

<u>Claim 2</u> $\forall f \in L^1(\mathbb{C}) \cap L^2(\mathbb{C})$, $\forall t > 0$, $\exists$ countably many closed squares Q_i as in Claim 1 s.t.

- $t \cdot \mathrm{vol}(Q_i) < \|f\|_{L^1(Q_i)} \leq 4t \cdot \mathrm{vol}(Q_i)$, $\forall i$;
- $|f(x)| \leq t$ a.e. on $\mathbb{C} \setminus Q$, $Q := \bigcup_{i=1}^{+\infty} Q_i$.

(left as an exercise).

<u>Claim 3</u> Conclusion of the proof.

Let $Q = \bigcup_{i=1}^{+\infty} Q_i$ as in Claim 2. Then $t \cdot \mathrm{vol}(Q) \leq \|f\|_1$.

We set

$$g(x) := \begin{cases} f(x) & x \notin Q, \\ \frac{1}{\mathrm{vol}(Q_i)} \cdot \int_{Q_i} f & x \in Q_i, \end{cases} \qquad h := f - g.$$

Obviously,

$$\|g\|_1 \leq \|f\|_1, \quad \|h\|_1 \leq 2 \cdot \|f\|_1,$$

$h \equiv 0$ on $\mathbb{C} \setminus Q$ and $\int_{Q_i} h = 0$, $\forall i$. $\overset{\text{Claim 1}}{\Longrightarrow}$ $\exists\, c > 0$ s.t.

$$\mu(t, Th) \leq c \cdot \left(\mathrm{vol}(Q) + \frac{1}{t} \|h\|_1 \right) \leq \frac{3c}{t} \cdot \|f\|_1.$$

Since by Claim 2 we have

$$|g(x)| \leq 4t \qquad \text{a.e. in } \mathbb{C}$$

and hence

$$\mu(t, Tg) \overset{\text{Lemma}}{\leq} \frac{1}{t^2} \cdot \|Tg\|_2^2 \overset{\text{Step 1}}{\leq} \frac{\|T\|^2}{t^2} \|g\|_2^2 \leq \frac{64}{t} \|g\|_1 \leq \frac{64}{t} \|f\|_1$$

and hence

$$\mu(2t, Tf) \leq \mu(t, Tg) + \mu(t, Th)$$
$$\leq \frac{64}{t}\|f\|_1 + \frac{3C}{t}\|f\|_1$$
$$= \frac{64 + 3C}{t}\|f\|_1 . \quad \checkmark$$

■