Reading course in Floer homology 25th May 2023

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The Conley-Zehnder index

References: • Jean Gutt "Conley-Zehnder index for path of symp. matrices"
• Audin & Damian "Morse theory and Floer homology"

<u>Goal</u>: associate an index to a periodic orbit

1. Associate a path of matrices to a periodic orbit

Let $x: S^1 \longrightarrow M$ be a non-degenerate, contractible, periodic orbit ($x(t) = \varphi_t(x(0))$).
Define:

$$\mathcal{S}^* := \{ \phi \in \mathcal{C}^0([0,1], Sp(2n)) \mid \phi(0) = id, \det(\phi(1) - id) \neq 0 \}$$

We want to associate to x a path of matrices in $\mathcal{S}^*$.

Let $u: D^2 \longrightarrow M$ be a capping of x. Over D^2, all symplectic bundles can be trivialized: $u^*TM \cong D^2 \times \mathbb{R}^{2n}$, and the trivializations are all homotopic.
We can choose a symplectic frame $(z_1(t), \dots, z_{2n}(t))$ along x. Then, in this symplectic frame, we have a path of matrices $t \longmapsto \varphi(t) \in \mathcal{S}^*$ representing $(d\varphi_t)_{x(0)}$.

<u>Claim</u>: φ is unique up to homotopy

↳ take two cappings $u, v: D^2 \longrightarrow M$ of x, and consider the map $w: S^2 \to M$ obtained by gluing u and v along the boundary. Consider the symp. trivialization $w^*TM \cong S^2 \times \mathbb{R}^{2n}$. By restricting this to one of the two hemispheres (e.g. the one relative to u), we get a different trivialization of u^*TM. But all trivializations are homotopic. Analogous for v^*TM.
By the transitivity property, we get the claim.

$S^2 \cong D \sqcup$

2. Construction of the Conley-Zehnder index.

Now we want to construct a map $\mu_{CZ}: \mathcal{S}^* \longrightarrow \mathbb{Z}$, which is uniquely determined by the following properties:

(HTP-homotopy): μ_{CZ} is constant on the connected components of $\mathcal{S}^*$;

(LOOP): ϕ loop in $Sp(2n)$, $\varphi \in \mathcal{S}^* \Rightarrow \mu_{CZ}(\phi\varphi) = \mu_{CZ}(\varphi) + 2\mu(\phi)$,
↳ Maslov index

(SGN-Signature): $S = S^t$, $\|S\| < 2\pi$ (i.e., all eigenvalues have absolute value $< 2\pi$) and $\varphi(t) = \exp(J_0 S t) \Rightarrow \mu_{CZ}(\varphi) = \text{Ind}(S) - n$.

<u>Recall: Maslov index</u>

Let $A \in Sp(2n)$. There exist a unique decomposition $A = OP$, where

- $O \in Sp(2n) \cap O(2n) \cong U(n) \ni X + iY \Rightarrow \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$;
- $P \in Sp(2n) \cap Sym^+(2n)$.

Moreover, there exists a unique $S \in sp(2n) \cap Sym(2n) =: V$ such that
$P = \exp(S)$.

↳ $Sp(2n) \cong V \times U(n)$
$\exp(S) \cdot O \mapsto (S, O)$

It follows that $U(n) \subset Sp(2n)$ is a deformation retract of $Sp(2n)$.

$$\cancel{\begin{array}{rcl} \pi : Sp(2n) & \longrightarrow & U(n) \\ P \cdot \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} & \longmapsto & \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \end{array}}$$

In particular, $\pi_1(Sp(2n)) \cong \pi_1(U(n)) \cong \mathbb{Z}$.

Let ϕ be a loop in $Sp(2n)$. We have:

$$\begin{tikzcd} I \arrow[r, "\phi"] & Sp(2n) \arrow[d, "\pi"] \\ & U(n) \arrow[r, "\det_{\mathbb{C}}"'] & S^1 \end{tikzcd}$$

where π denotes the retraction

$$\begin{array}{rcl} \pi : Sp(2n) & \longrightarrow & U(n) \\ P \cdot \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} & \longmapsto & \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \end{array}$$

We define the <u>MASLOV INDEX</u> of ϕ as:

$$\mu(\phi) = \deg(t \longmapsto \det(\pi(\phi(t)))).$$

※

<u>The rotation map ρ</u> (continuous extension of $U(n) \xrightarrow{\det_{\mathbb{C}}} S^1$)

Theorem: For (V, ω) symplectic, there exists a unique continuous map

$$\rho : Sp(V) \longrightarrow S^1$$

(it induces an isomorphism of fundamental groups)

such that:

(i) $T : V \longrightarrow \tilde{V}$ symp. $\Rightarrow \rho(TAT^{-1}) = \rho(A)$;

(ii) $(V, \omega) = (V_1 \times V_2, \omega_1 \times \omega_2) \Rightarrow \rho(A) = \rho(A_1)\rho(A_2)$;

(iii) $A = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in Sp(2n) \cap O(2n) \Rightarrow \rho(A) = \det_{\mathbb{C}}(X + iY)$;

(iv) A has no eigenvalue in $S^1 \Rightarrow \rho(A) = \pm 1$.

Define:

- $Sp(2n)^* := \{ A \in Sp(2n) \mid \det(A - id) \neq 0 \}$
- $\Sigma := Sp(2n) \setminus Sp(2n)^*$

<u>Lemma</u>: $Sp(2n)^*$ has two connected components:

path-connected

$$Sp(2n)^{\pm} := \{ A \mid \det(A - id) \gtrless 0 \}$$

and their inclusions in $Sp(2n)$ are trivial maps in π_1.

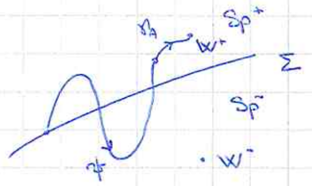
$$\begin{tikzcd} & & \mathbb{R} \arrow[d] \\ Sp(2n)^{\pm} \arrow[urr] \arrow[r, hook] & Sp(2n) \arrow[r, "\rho"'] & S^1 \end{tikzcd}$$

$$\begin{tikzcd} & & * \arrow[d] \\ \pi_1(Sp(2n)^{\pm}) \arrow[urr] \arrow[r, hook] & \pi_1(Sp(2n)) \arrow[r] & \mathbb{Z} \end{tikzcd}$$

We fix $W^\pm \in Sp(2n)^\pm$ such that $\rho(W^\pm) \in \{\pm 1\}$:

- $W^+ := -\mathrm{id} \implies \rho(W^+) = \left(\rho\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\right)^n = (-1)^n$
- $W^- := \mathrm{diag}(2,-1,\dots,-1,\frac{1}{2},-1,\dots,-1) \implies \rho(W^-) = \rho\begin{pmatrix}2&0\\0&1/2\end{pmatrix}\left(\rho\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\right)^{n-1} = (-1)^{n-1}$

For each $A \in Sp(2n)$, assign a path $\gamma_A \in C^0([0,1], Sp(2n)^*)$ such that $\gamma_A(1) \in \{W^\pm\}$. (take $\gamma_A \in Sp^+$ if $\psi(1) \in Sp^+$, otherwise $\gamma_A \in Sp^-$)



We consider the extension of ψ:

$$\tilde\psi = \psi * \gamma_{\psi(1)}$$

The <u>CONLEY-ZEHNDER INDEX</u> of $\psi \in \mathcal{S}^*$ is:

$$\mu_{CZ}(\psi) := \deg\left(t \longmapsto \rho^2(\tilde\psi(t))\right)$$

$\left(\begin{array}{l}\rho\circ\tilde\psi : [0,2] \longrightarrow S^1, \\ \rho(\tilde\psi(2)) = \rho(W^\pm) = \pm 1 \\ \implies \rho^2\circ\tilde\psi \text{ is a loop}\end{array}\right)$

<u>Remark</u>: μ_{CZ} is independent on the choice of γ_A. *

<u>Remark</u>: ρ is homotopic to $\det_{\mathbb{C}} \circ r \implies \mu_{CZ}(\psi) = \deg(\det^2_{\mathbb{C}} \circ r \circ \tilde\psi)$. *

<u>Theorem</u>: μ_{CZ} satisfies (HTP), (LOOP), (SGN) and the following properties:

(i) $\phi \in C^0([0,1], Sp(2n)) \implies \mu_{CZ}(\phi\psi\phi^{-1}) = \mu_{CZ}(\psi)$;

(ii) $\overset{\to\ \dim = 2n_1}{\psi_1 \in \mathcal{S}_1^*}$, $\overset{\to\ \dim = 2n_2}{\psi_2 \in \mathcal{S}_2^*} \implies \mu_{CZ}(\psi_1 \oplus \psi_2) = \mu_{CZ}(\psi_1) + \mu_{CZ}(\psi_2)$;

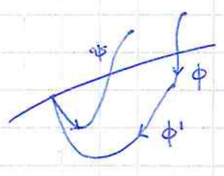
(iii) $\psi(s)$ has no eigenvalue on S^1 for $s > 0 \implies \mu_{CZ}(\psi) = 0$;

(iv) $(-1)^{\mu_{CZ}(\psi) - n} = \mathrm{sgn}(\det(\psi(1) - \mathrm{id}))$;

(v) $\mu_{CZ}(\psi^{-1}) = \mu_{CZ}(\psi^t) = -\mu_{CZ}(\psi)$.

Proof:

(i) Let $\tilde\psi$ be an extension of ψ. We need to find an extension of $\phi\psi\phi^{-1}$.



Let ϕ' be the path linking $\phi(1)$ to id (i.e., $\phi'(1) = \mathrm{id}$). We choose the extension:

$$\widetilde{\phi\psi\phi^{-1}} = \phi'\tilde\psi\phi'^{-1}$$

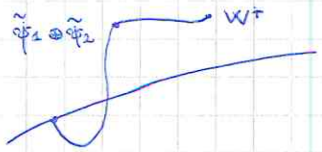
$$\implies \mu_{CZ}(\phi\psi\phi^{-1}) = \deg(\rho^2 \circ \phi'\tilde\psi\phi'^{-1}) \underset{\text{prop. (i) of } \rho}{=} \deg(\rho^2 \circ \tilde\psi) = \mu_{CZ}(\psi)$$

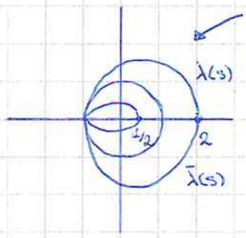
(ii) $\psi_1(1) \in Sp^-(n_1)$, $\psi_2(1) \in Sp^-(n_2)$ (the other cases are clear)

We need to choose the right symplectic basis to represent $\tilde\psi_1 \oplus \tilde\psi_2(2)$. With respect to the basis $(e_1, f_1, \dots, e_n, f_n)$ of $\mathbb{R}^{2(n_1+n_2)} = \mathbb{R}^{2n}$:

$$W' := \tilde\psi_1 \oplus \tilde\psi_2(1) = \left(\begin{array}{cccc|ccc} 2 & & & & & & \\ & 1/2 & & & & & \\ & & 2 & & & & \\ & & & 1/2 & & & \\ \hline & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \end{array}\right) \in Sp(2n)^+$$

So, $\tilde{\psi}_1 \oplus \tilde{\psi}_2(2) \in Sp(2n)^+$, but is different from $W^+ = -id$.
We can find a path of matrices in $Sp(2n)^+$ joining W^- to W^+, and on which ρ is constantly 1.





the positive eigenvs. are sent to -1

Since the value of ρ doesn't change we have:

$$\deg(\rho^2 \circ \tilde{\psi}_1 \oplus \tilde{\psi}_2) = \deg((\rho^2 \circ \tilde{\psi}_1)\cdot(\rho^2 \circ \tilde{\psi}_2)) = \deg(\rho^2 \circ \tilde{\psi}_1) + \deg(\rho^2 \circ \tilde{\psi}_2)$$

(HTP): Let $\psi_0 \cong \psi_1$ homotopic in $\mathcal{S}^*$.
We can define a homotopy between the extension $\tilde{\psi}_0$ and $\tilde{\psi}_1$.
Since ρ^2 is continuous and the degree map is invariant under homotopy, we have:

$$\mu_{CZ}(\psi_0) = \deg(\rho^2 \circ \tilde{\psi}_0) = \deg(\rho^2 \circ \tilde{\psi}_1) = \mu_{CZ}(\psi_1)$$

(LOOP): To prove this property we need the following fact:

<u>Fact</u>: ϕ loop, $\psi \in \mathcal{S}^*$, $\phi\psi \simeq \phi * \psi \implies \widetilde{\phi\psi} = \phi * \tilde{\psi}$

$$\mu_{CZ}(\phi\psi) = \deg(\rho^2 \circ (\phi * \tilde{\psi})) = \deg(\rho^2 \circ \phi * \rho^2 \circ \tilde{\psi}) = \deg(\rho^2 \circ \phi) + \deg(\rho^2 \circ \tilde{\psi}) =$$

$$= 2\mu(\phi) + \mu_{CZ}(\psi)$$

(SGN): $S = S^t$ can be orthogonally diagonalized and we can reduce to the case of S diagonal:

$$P_\lambda \in O(2n),\ \lambda \in [0,1],\ \text{with } P_0 = id \text{ and } S_\lambda = P_1 S P_1^t = \operatorname{diag}(a_1, \dots, a_{2n})$$

$$\|S\| < 2\pi \implies \|J_0 S\| < 2\pi \implies \exp(J_0 S) \text{ has no eigenvalue } 1$$

$$\implies t \longmapsto \exp(t J_0 S_\lambda) \in \mathcal{S}^*$$

Therefore, we can assume w.l.o.g. $S = \operatorname{diag}(\varepsilon, \dots, \varepsilon, -\varepsilon, \dots, -\varepsilon)$.

$$\implies \exp(tJ_0S) = \exp\left(t \left(\begin{array}{c|c} & \begin{matrix}\pm\varepsilon & & \\ & \ddots & \\ & & \pm\varepsilon\end{matrix} \\ \hline \begin{matrix}\pm\varepsilon & & \\ & \ddots & \\ & & \pm\varepsilon\end{matrix} & \end{array}\right)\right) \underset{\uparrow}{=} \exp\left(t \begin{pmatrix} \begin{matrix} 0 & \pm\varepsilon \\ \pm\varepsilon & 0 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} 0 & \pm\varepsilon \\ \pm\varepsilon & 0 \end{matrix} \end{pmatrix}\right) = \oplus \exp\left(t \begin{pmatrix} 0 & \pm\varepsilon \\ \pm\varepsilon & 0 \end{pmatrix}\right)$$

up to conjugation (not relevant because of prop. (i))

We only need to consider the paths $t \overset{\psi_1}{\longmapsto} \exp\begin{pmatrix} 0 & t\varepsilon \\ t\varepsilon & 0 \end{pmatrix}$, $t \overset{\psi_2}{\longmapsto} \exp\begin{pmatrix} 0 & -t\varepsilon \\ -t\varepsilon & 0 \end{pmatrix}$ and $t \overset{\psi_3}{\longmapsto} \exp\begin{pmatrix} 0 & -t\varepsilon \\ t\varepsilon & 0 \end{pmatrix}$. One shows, for instance:

$$\psi_1(t) = \begin{pmatrix} \cos\varepsilon t & \sin\varepsilon t \\ -\sin\varepsilon t & \cos\varepsilon t \end{pmatrix} \implies \rho(\psi_1(t)) = e^{-i\varepsilon t} \implies \rho(\tilde{\psi}_1(t)) \simeq e^{-i\pi t}$$

$$\implies \mu_{CZ}(\psi_1) = -1 = \operatorname{Ind}\begin{pmatrix} t\varepsilon & 0 \\ 0 & t\varepsilon \end{pmatrix} - 1.$$

Similarly for ψ_2 and ψ_3. Then use the additivity property (ii). $\square$

3. Alternative definition of Conley-Zehnder index

Let $\varphi \in \mathcal{S}^* \cap \mathcal{C}^1([0,1], Sp(2n))$ and define $S(t) := -J_0 \dot{\varphi}(t)\varphi^{-1}(t) \in \mathcal{C}^0([0,1], Sym(2n))$.
Then φ can be expressed as the solution of an ODE:

$$\begin{cases} \dot{\varphi}(t) = J_0 S \varphi \\ \varphi(0) = id \end{cases} \Rightarrow \varphi(t) = \exp\left(J \int_0^t S(\tau)\, d\tau\right)$$

Definition: • $t \in [0,1)$ is called CROSSING if $\det(\varphi(t) - id) = 0$.

• If t is a crossing, we can define the CROSSING FORM

$$\Gamma(\varphi, t) : \ker(\varphi(t) - id) \longrightarrow \mathbb{R}$$
$$v \longmapsto \omega_0(v, \dot{\varphi}(t) v)$$
$$\omega_0(v, \dot{\varphi}(t) v) = \omega_0(v, J_0 S(t)\underbrace{\varphi(t) v}_{=v}) = \langle S(t) v, v\rangle$$

• t is called REGULAR CROSSING if $S(t)$ is invertible.

Remark: Regular crossing are isolated. *

Theorem: For $\varphi \in \mathcal{S}^* \cap \mathcal{C}^1$ with only regular crossing

$$\mu_{CZ}(\varphi) = \frac{1}{2}\, sgn(\Gamma(\varphi, 0)) + \sum_{\substack{t>0 \\ \text{crossing}}} sgn(\Gamma(\varphi(t))$$

Remark: Each $\varphi \in \mathcal{S}^*$ is homotopic to $\varphi' \in \mathcal{S}^* \cap \mathcal{C}^1$ with regular crossings. *